Theorem: be able to apply

Theorem: and know what goes into the proof

Theorem: and be able to prove.

§3.3 Monotone Convergence Theorem. The least upper bound property is crucial. Understand Example 3.3.3(b).

§3.4 Theorem 3.4.4. Monotone Subsequence Theorem, Bolzano-Weierstrass Theorem (first proof).

§3.5 Know definition of Cauchy sequence. Cauchy Convergence Criterion. Proof uses Lemma: a Cauchy sequence is bounded; then Bolzano-Weierstrass to produce a candidate limit; then additional ε − δ argument to show that limit works.

Know definition of contractive sequence. A contractive sequence is a Cauchy sequence. Proof is straightforward once you use $a^k + a^{k+1} + \cdots + a^{k+\ell} = a^k (1 - a^{\ell+1})/(1 - a)$.

§3.6 Know definition of properly divergent sequence.

§4.1 Know definition of cluster point and of limit of a function at a cluster point. Theorem 4.1.5 (uniqueness of limit): basic and paradigmatic ε − δ argument. Sequential Criterion for Limits (Theorem 4.1.8). Divergence Criteria (4.1.9).

§4.2 Theorem 4.2.2 (f has a limit at c implies f is bounded on a neighborhood of c). Nice ε − δ argument. Theorem 4.2.3 on limits of sums, products, quotients. Theorem 4.2.6 on ≤-inequalities persisting to limits. Theorem 4.2.9 (lim_{x→c} f(x) > 0 implies that c has a δ-neighborhood on which f(x) > 0): useful theorem and illustrative proof.

§4.3 Not necessary to review for test. Check exercises.

§5.1 Know definition of “f continuous at c” in ε − δ terms. Understand Remark after Theorem 5.1.2. Sequential Criterion for Continuity. Know 5.1.6 Examples (g) and (h).
§5.2 Theorem 5.2.2 on sums, products and quotients of continuous functions. Theorem 5.2.6 (if $f$ continuous at $c$ and $g$ continuous at $f(c)$ then $g \circ f$ continuous at $c$): $\epsilon - \delta - \gamma$ argument.

§5.3 has three important theorems. Theorem 5.3.2 ($f$ continuous on $[a, b]$ implies $f$ bounded): proof by contradiction. Use $\mathbb{N}$ to construct a sequence, use Bolzano-Weierstrass to find a point where $f$ is not continuous. Maximum Theorem 5.3.4 ($f$ continuous on $[a, b]$ has “a maximum”: a point where it takes on its maximum value): use 5.3.2, the least upper bound axiom and $\mathbb{N}$ to define a sequence, and Bolzano-Weierstrass to extract a convergent subsequence; this identifies a candidate maximum; prove this point works. Same for minimum. “Location of Roots” Theorem ($f$ continuous on $[a, b]$ and $f(a) < 0 < f(b)$ implies $\exists c \in (a, b)$ with $f(c) = 0$): uses a bisection argument and the Nested Intervals Property. Intermediate Value Theorem is direct consequence. A consequence of these theorems is Theorem 5.3.9: if $f$ is continuous on $[a, b]$ then $f([a, b])$ is another closed interval.

§5.4 Know definition of uniform continuity and be able to show, for example, that $f(x) = 1/x$ on $(0, 1]$, which is continuous, is not uniformly continuous (this is discussed on pp. 136-137). Be familiar with the logical manipulations to get Nonuniformity criteria 5.4.2 (ii) and (iii). Uniform Continuity Theorem ($f$ continuous on $[a, b]$ is uniformly continuous): by contradiction using (iii) and Bolzano-Weierstrass to locate a point at which you can show $f$ is not continuous.

Know definition of Lipschitz function. A Lipschitz function is uniformly continuous.

Theorem 5.4.7 (a uniformly continuous function takes Cauchy sequences to Cauchy sequences): nice combination of Cauchy criterion with $\epsilon - \delta$ definition of uniform continuity. Continuous Extension Theorem 5.4.8 is a consequence.

§5.6 Here we will consider functions defined on an interval $I$ (without specifying which if any endpoints are included). Know distinction between “increasing” and “strictly increasing,” etc. and also “monotone” and “strictly monotone.” For $f$ increasing, understand the definition of the jump $j_f(c)$ of $f$ at an interior point $c$ of $I$ (it’s $\lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$) and the definitions of jumps at endpoints. Theorem 5.6.3 (An increasing $f$ is continuous on $I$ iff $j_f(c) = 0$ for every $c \in I$). And similarly for decreasing. Theorem 5.6.4 (a monotonic function on an interval $(a, b)$ has at most a
countable number of points of discontinuity): at most 1 with jump \( \geq (b-a) \), at most 2 with jump \( \geq (b-a)/2 \), etc., using 5.6.3. **Continuous Inverse Theorem 5.6.5** (a strictly monotone, continuous \( f \) defined on an interval \( I \) has a continuous inverse \( g \)): first \( g \) exists because \( f \) strictly monotonic; \( g \) is also (strictly) monotonic; a discontinuity of \( g \) would be a jump; this would force \( f \) to be missing a point.

§6.1 Here again \( f \) is defined on an interval \( I \). Know the definition of the derivative of \( f \) at \( c \in I \). **Theorem 6.1.2** (\( f \) has a derivative at \( c \) implies \( f \) continuous at \( c \)): directly from the definition, show \( \lim_{x \to c} (f(x) - f(c)) = 0 \). **Theorem 6.1.3 - Differentiation Rules** - pay attention to the quotient. **Carathéodory’s Theorem 6.1.5** (very useful in getting rid of troublesome denominators): proof is straightforward. **Chain Rule 6.1.6** - use Carathéodory. **Derivative of Inverse** - note requirement that \( f' \) is non-zero at \( c \); use Carathéodory.

§6.2 **Interior Extremum Theorem 6.2.1** (if \( c \) is an interior extremum of \( f \), then if \( f'(c) \) exists, it is 0): straightforward proof by contradiction, using definition of derivative. **Rolle’s Theorem 6.2.3** and **Mean Value Theorem 6.2.4** both for \( f \) continuous on \([a,b]\) and differentiable on \((a,b)\). RT: if \( f(a) = f(b) \) then there exists \( c \in (a,b) \) with \( f'(c) = 0 \). Use continuity and Maximum Theorem to find an extremum; show it must be interior; apply 6.2.1. MVT: there exists \( c \in (a,b) \) with \( f'(c) = (f(b) - f(a))/(b-a) \). Cook up a function expressing the difference between \( f \) and the straight-line function from \((a,f(a))\) to \((b,f(b))\), and apply Rolle’s Theorem. **Theorems 6.2.5 and 6.2.7** (with same hypotheses: \( f'(x) = 0 \) for all \( a < x < b \) iff \( f \) is constant; \( f'(x) \geq 0 \) for all \( a < x < b \) iff \( f \) is increasing; \( f'(x) \leq 0 \) for all \( a < x < b \) iff \( f \) is decreasing). Directly from MVT and definition of derivative. Note remark on p. 171 about \( f(x) = x^3 \), etc. **Darboux’s Theorem 6.2.12** (\( f \) differentiable on \([a,b]\) implies that \( f' \) takes on any value \( k \) between \( f'(a) \) and \( f'(b) \)) follows from **Lemma 6.2.11** (straightforward from definition of derivative) and the interior extremum theorem applied to \( g(x) = kx - f(x) \).

§6.4 Know the definition of the \( n \)th Taylor Polynomial \( P_n(x) \) approximating a function \( f \) at a point \( x_0 \). **Taylor’s Theorem 6.4.1** \( f(x) - P_n(x) = f^{(n+1)}(c)/(n+1)! (x - x_0)^{n+1} \) for some \( c \) between \( x_0 \) and \( x \): understand that it is proved by applying Rolle’s Theorem to an appropriately cooked up auxiliary function. **Newton’s Method 6.4.7**: understand how it works and why it gives “quadratic” convergence.