MAT 320 Fall 2007 Review for Midterm 2

- Theorem: be able to apply
- *Theorem*: and know what goes into the proof
- **Theorem**: and be able to prove.

§3.3 **Monotone Convergence Theorem**. The least upper bound property is crucial. Understand Example 3.3.3(b).

§3.4 Theorem 3.4.4. **Monotone Subsequence Theorem, Bolzano- Weierstrass Theorem (first proof)**.

§3.5 Know definition of Cauchy sequence. **Cauchy Convergence Criterion**. Proof uses **Lemma**: a Cauchy sequence is bounded; then Bolzano-Weierstrass to produce a candidate limit; then additional $\epsilon - \delta$ argument to show that limit works.

Know definition of contractive sequence. **A contractive sequence is a Cauchy sequence**. Proof is straightforward once you use $a^k + a^{k+1} + \cdots + a^{k+\ell} = a^k(1 - a^{\ell+1})/(1 - a)$.

§3.6 Know definition of properly divergent sequence.

§4.1 Know definition of cluster point and of limit of a function at a cluster point. **Theorem 4.1.5** (uniqueness of limit): basic and paradigmatic $\epsilon - \delta$ argument. *Sequential Criterion for Limits (Theorem 4.1.8).* Divergence Criteria (4.1.9).

§4.2 **Theorem 4.2.2** (f has a limit at c implies f is bounded on a neighborhood of c). Nice $\epsilon - \delta$ argument. *Theorem 4.2.3* on limits of sums, products, quotients. *Theorem 4.2.6* on $\leq$-inequalities persisting to limits. **Theorem 4.2.9** ($\lim_{x \to c} f(x) > 0$ implies that c has a δ-neighborhood on which $f(x) > 0$): useful theorem and illustrative proof.

§4.3 Not necessary to review for test. Check exercises.

§5.1 Know definition of "f continuous at c" in $\epsilon - \delta$ terms. Understand Remark after Theorem 5.1.2. **Sequential Criterion for Continuity**. Know 5.1.6 Examples (g) and (h).

§5.2 *Theorem 5.2.2* on sums, products and quotients of continuous functions. **Theorem 5.2.6** (if f continuous at c and g continuous at $f(c)$ then $g \circ f$ continuous at c): $\epsilon - \delta - \gamma$ argument.

§5.3 has three important theorems. **Theorem 5.3.2** (f continuous on $[a, b]$ implies f bounded): proof by contradiction. Use **N** to construct a sequence, use Bolzano-Weierstrass to find a point where f is not continuous. **Maximum Theorem 5.3.4** (f continuous on $[a, b]$ has "a maximum": a point where it takes on its maximum value): use 5.3.2, the least upper bound axiom and **N** to define a sequence, and Bolzano-Weierstrass to extract a convergent subsequence; this identifies a candidate maximum; prove this point works. Same for minimum. **"Location of Roots" Theorem** (f continuous on $[a, b]$ and $f(a) < 0 < f(b)$ implies $\exists c \in (a, b)$ with $f(c) = 0$): uses a bisection argument and the Nested Intervals Property. **Intermediate Value Theorem** is direct consequence. A consequence of these theorems is *Theorem 5.3.9*: if f is continuous on $[a, b]$ then $f([a, b])$ is another closed interval.

§5.4 Know definition of uniform continuity and be able to show, for example, that $f(x) = 1/x$ on $(0, 1]$, which is continuous, is not uniformly continuous (this is discussed on pp. 136-137). Be familiar with the logical manipulations to get Nonuniformity criteria 5.4.2 (ii) and (iii). **Uniform Continuity Theorem** (f continuous on $[a, b]$ is uniformly continuous): by contradiction using (iii) and Bolzano- Weierstrass to locate a point at which you can show f is not continuous.

Know definition of Lipschitz function. *A Lipschitz function is uniformly continuous.*

Theorem 5.4.7 (a uniformly continuous function takes Cauchy sequences to Cauchy sequences): nice combination of Cauchy criterion with $\epsilon - \delta$ definition of uniform continuity. *Continuous Extension Theorem 5.4.8* is a consequence.

§5.6 Here we will consider functions defined on an interval I (without specifying which if any endpoints are included). Know distinction betweem "increasing" and "strictly increasing," etc. and also "monotone" and "strictly monotone." For f increasing, understand the definition of the jump $j_f(c)$ of f at an interior point c of I (it's $\lim_{x \to c+} f(x) - \lim_{x \to c-} f(x)$) and the definitions of jumps at endpoints. *Theorem 5.6.3* (An increasing f is continuous on I iff $j_f(c) = 0$ for every $c \in I$). And similarly for decreasing. **Theorem 5.6.4** (a monotonic function on an interval (a, b) has at most a countable number of points of discontinuity): at most 1 with jump $\geq (b-a)$, at most 2 with jump $\geq (b-a)/2$, etc., using 5.6.3. **Continuous Inverse Theorem 5.6.5** (a strictly monotone, continuous f defined on an interval I has a continuous inverse g): first g exists because f strictly monotonic; g is also (strictly) monotonic; a discontinuity of g would be a jump; this would force I to be missing a point.

§6.1 Here again f is defined on an interval I. Know the definition of the derivative of f at $c \in I$. **Theorem 6.1.2** (f has a derivative at c implies f continuous at c): directly from the definition, show $\lim_{x \to c}(f(x) - f(c)) = 0$. *Theorem 6.1.3 - Differentiation Rules* - pay attention to the quotient. **Carathéodory's Theorem 6.1.5** (very useful in getting rid of troublesome denominators): proof is straightforward. **Chain Rule 6.1.6** -use Carathéodory. *Derivative of Inverse* - note requirement that $f'(c) \neq 0$; use Carathéodory.

§6.2 **Interior Extremum Theorem 6.2.1** (if c is an interior extremum of f, then if $f'(c)$ exists, it is 0): straightforward proof by contradiction, using definition of derivative. **Rolle's Theorem 6.2.3 and Mean Value Theorem 6.2.4** both for f continuous on $[a, b]$ and differentiable on (a, b). RT: if $f(a) = f(b)$ then there exists $c \in (a, b)$ with $f'(c) = 0$. Use continuity and Maximum Theorem to find an extremum; show it must be interior; apply 6.2.1. MVT: there exists $c \in (a, b)$ with $f'(c) = (f(b)-f(a))/(b-a)$. Cook up a function expressing the difference between f and the straight-line function from $(a, f(a))$ to $(b, f(b))$, and apply Rolle's Theorem. *Theorems 6.2.5 and 6.2.7* (with same hypotheses: $f'(x) = 0$ for all $a < x < b$ iff f is constant; $f'(x) \geq 0$ for all $a < x < b$ iff f is increasing; $f'(x) \leq 0$ for all $a < x < b$ iff f is decreasing). Directly from MVT and definition of derivative. Note remark on p. 171 about $f(x) = x^3$, etc. *Darboux's Theorem 6.2.12* (f differentiable on $[a, b]$ implies that f' takes on any value k between $f'(a)$ and $f'(b)$) follows from *Lemma 6.2.11* (straightforward from definition of derivative) and the interior extremum theorem applied to $g(x) = kx - f(x)$.

§6.4 Know the definition of the nth Taylor Polynomial $P_n(x)$ approximating a function f at a point x_0. *Taylor's Theorem 6.4.1* ($f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ for some c between x_0 and x): understand that it is proved by applying Rolle's Theorem to an appropriately cooked up auxiliary function. *Newton's Method 6.4.7*: understand how it works and why it gives "quadratic" convergence.