More on SYZ with Relative Fourier-Mukai Transforms

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These are notes taken from a talk by Ruijie Yang in the Fall 2019 RTG Student Seminar at Stony Brook University. He was following a paper of Arinkin and Polishchuk: *Fukaya Category and Fourier Transform*.

1 Introduction

The main take-aways of the talk are the following. When we assume the SYZ conjecture, say, in the setting of mirror Calabi-Yau's X and $\check{X}$ with special lagrangian fibrations, then

1. Deformation of the symplectic structure of X corresponds to deformation of the complex structure of $\check{X}$.
2. There is a "conceptual" SYZ transform between $D^{\pi}(\mathcal{F}(X))$ and $D^b_{coh}(\check{X})$. That is, we get a functor which is defined on certain classes of Lagrangians. The importance of this SYZ transform, which is very much like a Fourier-Mukai transform, is that in HMS, we have no conjectures for how to construct and equivalence of categories, only that one should exist.

2 Setting

As usual, we have mirror CY's which have special Lagrangian fibrations over the same base. The fibers are not always smooth. To recall, a compact Calabi-Yau manifold X of $\dim_{\mathbb{C}} X = n$ is firstly, a Kähler manifold so it has a Kähler form ω. Moreover, it has trivial canonical line bundle which means that we have a holomorphic volume form $\Omega \in H^{n,0}(X)$. A Lagrangian L is **special** if the imaginary part of Ω is zero, when restricted to L: $\operatorname{im}\Omega|_L = 0$.

$$\begin{array}{ccccc} X & & & & \check{X} \\ & \searrow^{\mu} & & {}^{\check{\mu}}\swarrow & \\ & & B & & \end{array}$$

There is a theorem of Robert McLean from 1998: Let $\mathcal{M}$ be the moduli space of special Lagrangians of X. Then $\mathcal{M}$ is a smooth manifold of real dimension n. So $B \subset \mathcal{M}$. Also, if $[L] \in \mathcal{M}$, then $T_{[L]}\mathcal{M} \cong H^1(L, \mathbb{R}) \cong H^{n-1}(L, \mathbb{R})$.

3 Deformations

Observe that we may try deforming a special Lagrangian L by pushing it around with a vector field that lives on the normal bundle. So a deformation of L corresponds to $V \in C^{\infty}(\mathcal{N}_L, \mathbb{R})$.

But we want the deformation to preserve the special nature of L. First, define $\alpha := -\iota_V \omega \in \Omega^1(X)$ and $\beta := \iota_V \operatorname{im} \Omega \in \Omega^{n-1}(X)$. Ruijie claimed that V gives us a deformation that preserves the special condition of L if and only if $d\alpha = 0$ and $d\beta = 0$. It seems there were some doubts about whether this is correct.

Looking at McLean's paper, he says: A normal vector field V to a compact special Lagrangian submanifold L is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the corresponding 1-form $(JV)^\flat$ is closed and co-closed, i.e., harmonic. I think J is the complex structure here and $\flat$ means using the Kähler metric to produce a 1-form from a vector field canonically. So in fact, because of the compatibility of the Riemannian structure and the symplectic structure, $\alpha = (JV)^\flat$. This tells us that we only need to look at α and not β. Or rather, studying α should give us what we need for β.

We also note that $H^1(L, \mathbb{Z}) \subset H^1(L, \mathbb{R}) \cong T_{[L]}\mathcal{M}$. This gives rise to a definition: we say a manifold M has **integral affine structure** (IAS) if there exists $\Lambda \subset TM$ which is a local system of integral lattices. A local system is a locally constant sheaf. If I'm not mistaken, my picture is that of integral lattices inside of the fibers of a vector bundle. Locally on $U \subset M$, the tangent bundle looks like $U \times \mathbb{R}^n$ and Λ gives the same lattice on each $\mathbb{R}^n$. I could be very wrong. Anyhow, B has IAS.

Let's consider now a space with a sheaf $(X, \mathcal{F})$ and a map $f : X \to Y$. We define $R^k f_* \mathcal{F}$ to be a sheaf which we can define on open sets: $R^k f_* \mathcal{F}(U) = H^k(f^{-1}(U), \mathcal{F})$. It is sensible to study this on stalks as well. Let : $\mu : X \to B$. Then a stalk $(R^1 \mu_* \mathbb{R})_b = H^1(L_b; \mathbb{R}) \cong T_b B$. We then have $R^1 \mu_* \mathbb{Z} \subset R^1 \mu_* \mathbb{R} \cong TB \cong R^{n-1} \mu_* \mathbb{R} \supset R^1 \mu_* \mathbb{Z}$.

Claim: Studying the integral affine structure on B is enough to recover X and $\check{X}$.

Here is a proposition: If $\Lambda \subset TB$ is IAS, then TB/Λ has complex structure and T^*B/Λ^* has symplectic structure. Of course, locally, TB/Λ has complex structure but the IAS somehow gives "linear charts" which allow us to extend the complex structure globally.

Now, let take B with IAS and $\Lambda_1, \Lambda \subset TB$. We'll assume that we have canonical way of identifying $TB \cong T^*B$, say with a metric. Also, we'll assume $\Lambda_1 \cong \Lambda_2^*$ and $\Lambda_2 \cong \Lambda_1^*$. Let $X = TB/\Lambda \cong T^*B/\Lambda_2^*$; this is symplectic. And $\check{X} = T^*B/\Lambda_1^* = TB/\Lambda_2$. This is complex. Both are controlled by just Λ_2. Hence, by deforming the lattice Λ_2, we are able to deform X and $\check{X}$ at the same time.

4 Fourier-Mukai Transform

Let's first consider the absolute Fourier-Mukai transform. The setting is that of tori. We have a vector space V of real dimension n. Let $A = V/\Lambda$ and $\check{A} = V^*/\Lambda^*$. From previous talks, we say that the dual torus $\check{A}$ can be thought of as a space of pairs (L, ∇) on A. The connections ∇ here are classified by monodromy. Let (P, ∇^P) be something called the Poincaré line bundle on $A \times \check{A}$. Using the description of the dual torus as above, then we require $(P, \nabla^P)|_{A \times [L, \nabla]} \cong (L, \nabla)$.

Next, let $S \subset A$ be a finite set, $\{x_1, ..., x_r\}$ or something. Associate to each x_i a vector space V_i. This defines for us a skyscraper sheaf $\mathcal{F}$: $\mathcal{F}(U) = \bigoplus_{x \in U \cap S} V_x$. Let $Sky(A)$ be the category of skyscraper sheaves on A and $Loc(\check{A})$ be the category of local systems on $\check{A}$. The Fourier-Mukai transform is a functor $\Phi : Sky(A) \to Loc(\check{A})$. How do we define this? Let $\pi_1 : A \times \check{A} \to A$ and $\pi_2 : A \times \check{A} \to \check{A}$ be projection maps. Then $\Phi(\mathcal{F}) := (\pi_2)_*(\pi_1^* \mathcal{F} \otimes P)$. What does this mean? We pullback $\mathcal{F}$ to the product space, tensor with P, then push forward.

We can compare this to the usual Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

The P is like the $e^{-2\pi i x\xi}$ term. Recall that we're looking at monodromy of connections which are basically classified by $e^{i\theta}$. So it makes sense to consider P. Also, note that integration makes it so that the variable x no longer appears and we only have ξ. Similarly, pushing forward by π_2 kills off the variabls in A and leaves just $\check{A}$.

Let's look at a stalk over $a \in \check{A}$. $\Phi(\mathcal{F})_a = \pi_1^*\mathcal{F} \otimes P|_{\pi_2^{-1}(a)} = \pi_1^*\mathcal{F} \otimes P|_{A \times a} = \bigoplus_{x \in S} V_x \otimes P_{(x,a)}$. This is a vector space. If $\mathcal{F}$ is a skyscraper sheaf associated with $V_1, ..., V_r$, then $\Phi(\mathcal{F})$ is a vector bundle of rank $\sum \dim V_i$.

Impressive fact: $\Phi : Sky(A) \to Loc(\check{A})$ is an equivalence of categories! And in a proper enlargement of categories, the inverse equivalence is simply Φ, up to $\pm$, just like how the inverse Fourier transform replaces the minus sign in $e^{-2\pi i x \cdot \xi}$ with a plus sign.

Let us return to our setting then. We have X and $\check{X}$ fibered over B. Take a Lagrangian $L \subset X$ that is transverse to all the fibers and let $(\mathcal{E}, \nabla)$ be a local system on L. L could possibly intersect the fibers in many places. It's even possible that the number of intersections for different fibers changes. In any event, $\mathcal{E}|_{L \cap L_b}$ is a skyscraper sheaf on L_b. Then, applying the Fourier-Mukai transform to this sheaf, we get a vector bundle on $\check{L}_b := \check{\mu}^{-1}(b)$. We then glue along all $b \in B$ and get a vector bundle on $\check{X}$.

This is a pretty interesting construction because thus far, we don't really have good candidates for the equivalences of categories in HMS. HMS only conjectures the existence of the functor. This functor isn't defined on the full Fukaya category, only the Lagrangians transverse to fibers. Yet, it is a promising start. It turns out that in the case that X is a 2-torus, Φ is an equivalence of categories. I think Polishchuk and Zaslow's paper on HMS for elliptic curves was submitted in January 1998 and this paper of Arinkin and Polishchuk was submitted in November 1998.