

MAT 126 Calculus B Spring 2006 Practice Final Exam — Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values. This practice exam contains more problems than the actual test to give you more practice.

1. Evaluate the following definite integrals:

(a)

$$\int_1^9 \ln \sqrt{x} dx$$
$$\int_1^9 \ln \sqrt{x} dx = \frac{1}{2} \int_1^9 \ln x dx = \frac{1}{2} (x \ln x - x) \Big|_1^9$$
$$= \frac{1}{2} (9 \ln 9 - 9 - (\ln 1 - 1)) = 9 \ln 3 - 4,$$

where we have used that the antiderivative of $\ln x$ is $x \ln x - x$ (see Section 5.6, Example 2).

(b)

$$\int_0^2 \frac{x}{1+2x^2} dx$$

Using the substitution $u = 1 + 2x^2$ we get $du = 4xdx$, so that $xdx = \frac{1}{4}du$, and the limits of integration $x = 0$ and $x = 2$ correspond to $u = 1$ and $u = 9$. We get

$$\int_0^2 \frac{x}{1+2x^2} dx = \frac{1}{4} \int_1^9 \frac{du}{u}$$
$$= \frac{1}{4} \ln u \Big|_1^9 = \frac{1}{2} \ln 3.$$

(c)

$$\int_1^e \frac{(\ln x)^3}{x} dx$$

The substitution $u = \ln x$ gives $du = \frac{dx}{x}$ and limits of integration $x = 1$ and $x = e$ correspond to $u = 0$ and $u = 1$. We have

$$\int_1^e \frac{(\ln x)^3}{x} dx = \int_0^1 u^3 du = \frac{1}{4} u^4 \Big|_0^1 = \frac{1}{4}.$$

(d)

$$\int_{-1}^1 x^2 \sin(x^5) dx$$

The function $f(x) = x^2 \sin(x^5)$ is odd, $f(-x) = -f(x)$, so using the property of symmetric functions (see Section 5.5), we get

$$\int_{-1}^1 x^2 \sin(x^5) dx = 0.$$

(e)

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$$

Using the substitution $u = \sin^{-1} x$, we get $du = \frac{dx}{\sqrt{1-x^2}}$, and the limits of integration $x = 0$ and $x = 1/2$ correspond to $u = 0$ and $u = \pi/6$. We get

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \frac{1}{2} u^2 \Big|_0^{\pi/6} = \frac{\pi^2}{72}.$$

(f)

$$\int_1^4 \sqrt{t} \ln t dt$$

Here we use integration by parts with $u = \ln t$ and $dv = \sqrt{t}$. We have $du = \frac{dt}{t}$ and $v = \frac{2}{3}t^{3/2}$, so that

$$\begin{aligned} \int_1^4 \sqrt{t} \ln t dt &= \int_1^4 u dv = uv \Big|_1^4 - \int_1^4 v du \\ &= \frac{2}{3}(16 \ln 2) - \frac{2}{3} \int_1^4 \sqrt{t} dt \\ &= \frac{32}{3} \ln 2 - \frac{4}{9} t^{3/2} \Big|_1^4 = \frac{32}{3} \ln 2 - \frac{28}{9}. \end{aligned}$$

(g)

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$$

We use the substitution $u = 1 + 2x$, so that $du = 2dx$ and the limits of integration $x = 0$ and $x = 13$ correspond to $u = 1$ and $u = 27$. We get

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \frac{1}{2} \int_1^{27} u^{-2/3} du = \frac{1}{2} 3u^{1/3} \Big|_1^{27} = 3.$$

(h)

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$$

This is a trigonometric integral. Writing

$$\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x,$$

we recognize the substitution $u = \cos x$. We have $du = -\sin x \, dx$, and the limits of integration $x = 0$ and $x = \frac{\pi}{2}$ correspond to $u = 1$ and $u = 0$. We get

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = -\int_1^0 (1 - u^2) \, du = \int_0^1 (1 - u^2) \, du = \left(u - \frac{u^3}{3} \right) \Big|_0^1 = \frac{2}{3}.$$

2. Evaluate the following indefinite integrals:

(a)

$$\int x^2 e^x \, dx$$

This is Example 3 in Section 5.6.

(b)

$$\int \frac{2x^3 + 1}{x^2 + 1} \, dx$$

Doing long division, or simplifying as follows:

$$\frac{2x^3 + 1}{x^2 + 1} = \frac{(2x^3 + 2x) + 1 - 2x}{x^2 + 1} = 2x + \frac{1 - 2x}{x^2 + 1},$$

we get

$$\begin{aligned} \int \frac{2x^3 + 1}{x^2 + 1} \, dx &= \int \left(2x + \frac{1 - 2x}{x^2 + 1} \right) \, dx \\ &= x^2 + \int \frac{1}{x^2 + 1} \, dx - 2 \int \frac{x}{x^2 + 1} \, dx \\ &= x^2 + \tan^{-1} x - \ln(x^2 + 1) + C, \end{aligned}$$

where in the last integral we have used the substitution $u = x^2 + 1$.

(c)

$$\int \frac{\tan^{-1} x}{1 + x^2} \, dx$$

Using the substitution $u = \tan^{-1} x$, we get $du = \frac{dx}{1 + x^2}$, so

that

$$\int \frac{\tan^{-1} x}{1 + x^2} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\tan^{-1} x)^2 + C.$$

(d)

$$\int \sin^{-1} x \, dx$$

Using integration by parts with $u = \sin^{-1} x$ and $dv = dx$ we get

$$du = \frac{dx}{\sqrt{1-x^2}} \quad \text{and} \quad v = x,$$

so that

$$\begin{aligned} \int \sin^{-1} x \, dx &= \int u \, dv = uv - \int v \, du \\ &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx. \end{aligned}$$

To compute the last integral, use the substitution $u = 1 - x^2$, so that $du = -2x dx$ and

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Thus, finally,

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

(e)

$$\int \frac{x-1}{x^2+3x+2} dx$$

Using partial fractions,

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

where A and B are such that

$$x-1 = A(x+2) + B(x+1)$$

holds for all x . Setting $x = -1$, we get $A = -2$, and setting $x = -2$, we get $B = 3$. Thus

$$\begin{aligned} \int \frac{x-1}{x^2+3x+2} dx &= \int \left(-\frac{2}{x+1} + \frac{3}{x+2} \right) dx \\ &= -\log(x+1)^2 + \log|x+2|^3 + C. \end{aligned}$$

(f)

$$\int t^2 \cos(1-t^3) dt$$

Using the substitution $u = 1 - t^3$ we get $du = -3t^2 dt$ and

$$\int t^2 \cos(1-t^3) dt = -\frac{1}{3} \int \cos u du = -\frac{1}{3} \sin u + C = -\frac{1}{3} \sin(1-t^3) + C.$$

(g)

$$\int e^x \sqrt[3]{1+e^x} dx$$

Using the substitution $u = 1 + e^x$ we get $du = e^x dx$ and

$$\int e^x \sqrt[3]{1+e^x} dx = \int u^{\frac{1}{3}} du = \frac{3}{4} u^{\frac{4}{3}} + C = \frac{3}{4} (1+e^x)^{\frac{4}{3}} + C.$$

(h)

$$\int \cos^5 x dx$$

This is a trigonometric integral. Writing

$$\cos^5 x = \cos^4 x \cos x = (1 - \sin^2 x)^2 \cos x,$$

we recognize the substitution $u = \sin x$. We have $du = \cos x dx$ and

$$\begin{aligned} \int \cos^5 x dx &= \int (1 - \sin^2 x)^2 \cos x dx = \int (1 - u^2)^2 du = \int (1 - 2u^2 + u^4) du \\ &= u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C. \end{aligned}$$

3. (a) Write a formula for $\cos^2 x$ in terms of $\sin^2 x$.

$$\cos^2 x = 1 - \sin^2 x.$$

(b) Evaluate

$$\int \cos^3 x \sin^2 x dx$$

$$\int \cos^3 x \sin^2 x dx = \int (1 - \sin^2 x) \sin^2 x \cos x dx,$$

and using the substitution $u = \sin x$, we get $du = \cos x dx$ and

$$\begin{aligned} \int \cos^3 x \sin^2 x dx &= \int u^2(1-u^2) du = \int (u^2 - u^4) du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C. \end{aligned}$$

4. Let

$$f(x) = \int_2^{\sqrt{x}} \frac{\sin t}{t} dt + x^2$$

(a) Find $f'(x)$.

Using the chain rule with $u = \sqrt{x}$ and the Fundamental Theorem of Calculus, we get

$$\begin{aligned} \frac{df}{dx}(x) &= \frac{d}{du} \left(\int_2^u \frac{\sin t}{t} dt \right) \frac{du}{dx} \Big|_{u=\sqrt{x}} + 2x \\ &= \frac{\sin \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2x = \frac{\sin \sqrt{x}}{2x} + 2x. \end{aligned}$$

(b) Evaluate $f(4)$.

$$f(4) = \int_2^{\sqrt{4}} \frac{\sin t}{t} dt + 4^2 = \int_2^2 \frac{\sin t}{t} dt + 16 = 16.$$

5. Find a function f and a number a such that for x ,

$$1 + \int_a^x tf(t) dt = x^3$$

Setting in the equation $x = a$, we get $1 = a^3$, so that $a = 1$. Differentiating both sides of the equation with respect to x and using the Fundamental Theorem of Calculus, we get

$$xf(x) = 3x^2,$$

so that $f(x) = 3x$.

6. (a) Let

$$I = \int_0^4 e^{x^2} dx$$

For any value of n list the numbers L_n, R_n, M_n, T_n and I in increasing order.

The function $f(x) = e^{x^2}$ is increasing and concave upward on the real line (check it using the second derivative test). From the graph (sketch it!) we get

$$L_n < M_n < I < T_n < R_n.$$

(Here we used the analog of Fig. 5 on p. 419 (sketch it!) to get the relation $M_n < I < T_n$).

(b) Repeat part (a) for

$$I = \int_0^{\sqrt{2}/2} e^{-x^2} dx$$

The function $f(x) = e^{-x^2}$ is decreasing and concave downward when $0 \leq x \leq \sqrt{2}/2$ (check it using the second derivative test) . From the graph and analog of Fig. 5 (sketch them!) we get

$$R_n < T_n < I < M_n < L_n.$$

7. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a)

$$\int_0^{\infty} e^{-x} dx$$

The integral $\int_0^t e^{-x} dx$ exists for every number $t \geq 0$ and

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t e^{-x} dx &= \lim_{t \rightarrow \infty} (-e^{-x}) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} (1 - e^{-t}) = 1. \end{aligned}$$

The improper integral of Type 1 is convergent and

$$\int_0^{\infty} e^{-x} dx = 1.$$

(b)

$$\int_0^1 \frac{1}{\sqrt{x}} dx$$

The integral $\int_t^1 \frac{1}{\sqrt{x}} dx$ exists for every number $t > 0$ and

$$\begin{aligned} \lim_{t \rightarrow 0} \int_t^1 \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow 0} (2\sqrt{x}) \Big|_t^1 \\ &= 2 \lim_{t \rightarrow 0} (1 - \sqrt{t}) = 2. \end{aligned}$$

The improper integral of Type 2 is convergent and

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2.$$

(c)

$$\int_0^3 \frac{1}{x\sqrt{x}} dx$$

The integral $\int_t^3 \frac{1}{x\sqrt{x}} dx$ exists for every number $t > 0$ and

$$\begin{aligned} \lim_{t \rightarrow 0} \int_t^3 \frac{1}{x\sqrt{x}} dx &= \lim_{t \rightarrow 0} (-2x^{-1/2}) \Big|_t^3 \\ &= -\frac{2}{\sqrt{3}} + 2 \lim_{t \rightarrow 0} \frac{1}{\sqrt{t}}. \end{aligned}$$

The last limit is ∞ — does not exist as a finite number, so that improper integral of Type 2 is divergent.

(d)

$$\int_{-\infty}^{\infty} xe^{-x^2} dx$$

Since

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^0 xe^{-x^2} dx + \int_0^{\infty} xe^{-x^2} dx,$$

we must evaluate both integrals separately. Using the substitution $u = x^2$ (one could also use $u = -x^2$), we get $du = 2xdx$ and

$$\begin{aligned} \int_0^{\infty} xe^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t xe^{-x^2} dx = \frac{1}{2} \lim_{t \rightarrow \infty} \int_0^{t^2} e^{-u} du \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} (-e^{-u}) \Big|_0^{t^2} \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} (1 - e^{-t^2}) = \frac{1}{2}. \end{aligned}$$

Using this result and the fact that xe^{-x^2} is an odd function we get

$$\begin{aligned} \int_{-\infty}^0 xe^{-x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 xe^{-x^2} dx \\ &= - \lim_{-t \rightarrow \infty} \int_0^{-t} xe^{-x^2} dx = -\frac{1}{2}. \end{aligned}$$

Thus the improper integral of Type 1 is convergent and

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

(e)

$$\int_0^1 \frac{1}{4y-1} dy$$

The integrand is $f(y) = \frac{1}{4y-1}$, and it is discontinuous (blows up) at $y = \frac{1}{4}$. Thus

$$\int_0^1 \frac{1}{4y-1} dy = \int_0^{\frac{1}{4}} \frac{1}{4y-1} dy + \int_{\frac{1}{4}}^1 \frac{1}{4y-1} dy,$$

and we need to investigate both improper integrals of Type 2. For the first integral we have, using the substitution $u = 4y - 1$, $du = 4dy$,

$$\begin{aligned} \int_0^{\frac{1}{4}} \frac{1}{4y-1} dy &= \lim_{t \rightarrow \frac{1}{4}^-} \int_0^t \frac{1}{4y-1} dy = \frac{1}{4} \lim_{t \rightarrow \frac{1}{4}^-} \int_{-1}^{4t-1} \frac{1}{u} du \\ &= \frac{1}{4} \lim_{t \rightarrow \frac{1}{4}^-} \ln |u| \Big|_{-1}^{4t-1} = \frac{1}{4} \lim_{t \rightarrow \frac{1}{4}^-} (\ln |4t-1| - \ln |-1|) \\ &= \frac{1}{4} \lim_{t \rightarrow \frac{1}{4}^-} \ln |4t-1| = -\infty, \end{aligned}$$

since $\ln 0 = -\infty$. Thus the first improper integral is divergent, so that the integral in question is also divergent.

8. Find the area of the region bounded by the curves:

(a) $y = x^2$ and $y = x^4$.

The curves intersect at the points $x = -1, 0, 1$ and the top and bottom boundaries of the enclosed region are $y = x^2$ and $y = x^4$ (sketch the graph!). We have

$$\begin{aligned} A &= \int_{-1}^1 (x^2 - x^4) dx = 2 \int_0^1 (x^2 - x^4) dx \\ &= 2 \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{4}{15}. \end{aligned}$$

(b) $x + y^2 = 2$ and $x + y = 0$.

The curves intersect at the points with coordinates $(-2, 2)$ and $(1, -1)$, and the top and bottom boundaries of the enclosed region are $x = 2 - y^2$ and $x = -y$, where we are using y as an independent variable (sketch the graph!). We have

$$\begin{aligned} A &= \int_{-1}^2 (2 - y^2 - (-y)) dy = \int_{-1}^2 (2 - y^2 + y) dy \\ &= \left(2y - \frac{y^3}{3} + \frac{y^2}{2} \right) \Big|_{-1}^2 = 4\frac{1}{2}. \end{aligned}$$

9. (a) Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y^2 = x$ about the x -axis.

The curves $y = x^2$ and $y = \sqrt{x}$ (we solved the second equation for y , which is assumed to be positive) intersect at $x = 0$ and $x = 1$. The region has the curve $y = \sqrt{x}$ as the top boundary and the curve $y = x^2$ as the bottom boundary (sketch the graph!). A cross-section is a washer with the inner radius x^2 and the outer radius \sqrt{x} . The cross-sectional area is $A(x) = \pi(x - x^4)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_0^1 A(x) dx = \pi \int_0^1 (x - x^4) dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{3\pi}{10}. \end{aligned}$$

- (b) Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sec x$, $y = 1$, $x = -1$ and $x = 1$ about the x -axis.

The region has the horizontal line $y = \sec x$ as the top boundary, the curve $y = 1$ as the bottom boundary, and the lines $x = -1$ and $x = 1$ as the vertical boundaries (sketch the graph!). A cross-section is a washer with the inner radius 1 and the outer radius $\sec x$. The cross-sectional area is $A(x) = \pi(\sec^2 x - 1)$, and the volume of the solid of revolution is

$$\begin{aligned} V &= \int_{-1}^1 A(x) dx = \pi \int_{-1}^1 (\sec^2 x - 1) dx = 2\pi \int_0^1 (\sec^2 x - 1) dx \\ &= 2\pi (\tan x - x) \Big|_0^1 = 2\pi(\tan 1 - 1). \end{aligned}$$

10. Find the length of the following curves:

- (a) $y = x^{3/2}$, $0 \leq x \leq 2$.

$$L = \int_0^2 \sqrt{1 + (y')^2} dx = \int_0^2 \sqrt{1 + \frac{9}{4}x} dx.$$

Using the substitution $u = 1 + \frac{9}{4}x$, we get

$$\begin{aligned} L &= \frac{4}{9} \int_1^{11/2} \sqrt{u} du \\ &= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{11/2} = \frac{8}{27} \left(\left(\frac{11}{2} \right)^{3/2} - 1 \right). \end{aligned}$$

(b)

$$y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \leq x \leq 2$$

$$L = L = \int_1^2 \sqrt{1 + (y')^2} dx = \int_1^2 \sqrt{1 + \frac{1}{4} \left(x - \frac{1}{x}\right)^2} dx.$$

We have, by simple algebra,

$$\begin{aligned} 1 + \frac{1}{4} \left(x - \frac{1}{x}\right)^2 &= 1 + \frac{1}{4}(x^2 - 2 + x^{-2}) = \frac{1}{4}(x^2 + x^{-2}) + \frac{1}{2} \\ &= \frac{1}{4}(x^2 + 2 + x^{-2}) = \frac{1}{4}(x + x^{-1})^2. \end{aligned}$$

Thus

$$L = \int_1^2 \sqrt{\frac{1}{4}(x + x^{-1})^2} dx = \frac{1}{2} \int_1^2 (x + x^{-1}) dx = \frac{1}{2} \left(\frac{x^2}{2} + \ln x \right) \Big|_1^2 = \frac{3}{4} + \frac{\ln 2}{2}.$$

11. Find the average value f_{ave} of f on the given interval.

(a) $f(x) = x \sin(x^2)$ on $[0, \sqrt{\pi}]$.

We get, using the substitution $u = x^2$, $du = 2x dx$,

$$\begin{aligned} f_{ave} &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} x \sin(x^2) dx = \frac{1}{2\sqrt{\pi}} \int_0^{\pi} \sin u du \\ &= \frac{1}{2\sqrt{\pi}} (-\cos u) \Big|_0^{\pi} = \frac{1}{2\sqrt{\pi}} (-\cos \pi + \cos 0) = \frac{1}{\sqrt{\pi}}. \end{aligned}$$

(b) $f(x) = 4 - x^2$ on $[0, 3]$.

$$\begin{aligned} f_{ave} &= \frac{1}{3} \int_0^3 (4 - x^2) dx = \frac{1}{3} \left(4x - \frac{x^3}{3} \right) \Big|_0^3 \\ &= \frac{1}{3}(12 - 9) = 1. \end{aligned}$$

(c) For f as in part (b) find the number c in $[0, 3]$ such that

$f(c) = f_{ave}$.

Solving $4 - c^2 = 1$ we get $c = \sqrt{3}$ as the only solution which belongs to the interval $[0, 3]$.