Math 313 (Fall ’09)  

Homework 7  

due Nov 5  

NOTE: this Homework is worth 22 points (10 + 12)!

The following exercises refer to the textbook (the seventh edition).

• Ch16: 11, 13, 18, 21, 42  
• Ch17: 7, 11, 14, 23,  

2nd Part:  

1) i) Let $a$ be an element of a ring $R$. Assume that $a^4 = a^2$.  
   Prove that $a^{2n} = a^2$.  
   ii) With assumptions as above, is it true that $a^2 = 1$ or 0?  
   (if the answer is yes, prove it; if the answer is no, give a  
   counterexample)  
   iii) Is $\mathbb{Z}_6$ a subring of $\mathbb{Z}_{12}$? A subgroup?  

2) Suppose that $F$ is a field with 27 elements. Show that for every  
   element $a \in F$, $5a = -a$. (Hint: what is the characteristic of  
   $F$).  

3) i) Find all the ideals in $\mathbb{Z}_{12}$. Which one are prime, which one  
   are maximal?  
   ii) Give example of a prime ideal that is not maximal. Give  
   example of a maximal idea that is not prime.  
   iii)* Let $R$ be a finite ring (commutative with unit). Prove that  
   every prime ideal is maximal.  

4) i) Find all the morphism between $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$.  
   ii) Show that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are isomorphic as abelian  
   groups, but can not be isomorphic as rings.  
   iii) Assume that $\phi : D \rightarrow S$ is a non-zero morphism between  
   two domains. Prove that $\phi(1_D) = 1_S$. What can you say  
   about the characteristics of $D$ and $S$?  

5) i) Compute the addition and multiplication tables for $\mathbb{Z}_2[X]/\langle x^2 + x + 1 \rangle$.  
   ii) Give example of a polynomial of degree 3 in $\mathbb{Z}_6[X]$ such  
   that every element in $\mathbb{Z}_6$ is a root.  
   iii) Prove that $\mathbb{Q}[x]/\langle x^2 - 2 \rangle \cong \mathbb{Q}[\sqrt{2}]$. Prove in two ways that  
   $\mathbb{Q}[\sqrt{2}]$ is a field.  

6) Decide if $\mathbb{Z}_7[X]/\langle x^3 + x + 6 \rangle$ is a field or not.