Math 313 (Fall '09)

Homework 7

due Nov 5

NOTE: this Homework is worth 22 points (10 + 12)!

The following exercises refer to the textbook (the seventh edition).

- Ch16: 11, 13, 18, 21, 42
- Ch17: 7, 11, 14, 23,

2nd Part:

1) i) Let a be an element of a ring R. Assume that $a^4 = a^2$. Prove that $a^{2n} = a^2$.
   ii) With assumptions as above, is it true that $a^2 = 1$ or 0? (if the answer is yes, prove it; if the answer is no, give a counterexample)
   iii) Is $\mathbb{Z}_6$ a subring of $\mathbb{Z}_{12}$? A subgroup?
2) Suppose that F is a field with 27 elements. Show that for every element $a \in F$, $5a = -a$. (Hint: what is the characteristic of F).
3) i) Find all the ideals in $\mathbb{Z}_{12}$. Which one are prime, which one are maximal?
   ii) Give example of a prime ideal that is not maximal. Give example of a maximal idea that is not prime.
   iii)* Let R be a finite ring (commutative with unit). Prove that every prime ideal is maximal.
4) i) Find all the morphism between $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.
   ii) Show that $Q[\sqrt{2}]$ and $Q[\sqrt{3}]$ are isomorphic as abelian groups, but can not be isomorphic as rings.
   iii) Assume that $\phi : D \to S$ is a non-zero morphism between two domains. Prove that $\phi(1_D) = 1_S$. What can you say about the characteristics of D and S?
5) i) Compute the addition and multiplication tables for $\mathbb{Z}_2[X]/\langle x^2 + x + 1\rangle$.
   ii) Give example of a polynomial of degree 3 in $\mathbb{Z}_6[X]$ such that every element in $\mathbb{Z}_6$ is a root.
   iii) Prove that $\mathbb{Q}[x]/\langle x^2 - 2\rangle \cong \mathbb{Q}[\sqrt{2}]$. Prove in two ways that $\mathbb{Q}[\sqrt{2}]$ is a field.
6) Decide if $\mathbb{Z}_7[X]/\langle x^3 + x + 6\rangle$ is a field or not.