Problem 1. (i) Let $a$ be real, $0 \leq a < 1$. Let $U_a$ be the open set obtained from the unit disk $\{ |z| < 1 \}$ by removing the segment $[a, 1]$ of the real line. Construct a conformal isomorphism between $U_0$ and $U_a$ for $a > 0$.

(ii) Construct a conformal isomorphism between $U_0$ and the unit disk $\{ |z| < 1 \}$.

Problem 2. Let $f : D \to D$ be a holomorphic map of the unit disk $D = \{ |z| < 1 \}$ into itself. Show that for all $a \in D$,$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$Hint: Let $g$ be an automorphism of $D$ that maps $0$ to $a$, and $h$ an automorphism that maps $f(a)$ to $0$. Apply the Schwarz lemma to $F = h \circ f \circ g$.

Problem 3. (Blaschke products) A function of the form$$B(z) = \lambda \left( \frac{z - z_1}{1 - \overline{z_1}z} \right) \cdots \left( \frac{z - z_n}{1 - \overline{z_n}z} \right),$$where $|\lambda| = 1$ and $|z_j| < 1$ for all $1 \leq j \leq n$, is called a finite Blaschke product. Check that $B(z)$ is holomorphic in the closed unit disk $\{ |z| \leq 1 \}$, maps this disk into itself and has zeroes at $z_1, \ldots, z_n$.

Suppose that $f$ is holomorphic in the closed unit disk $\{ |z| \leq 1 \}$ and maps this disk into itself. Suppose also that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f$ must be a finite Blaschke product.

Problem 4. Let $f : D \to D$ be a holomorphic map of the unit disk $D = \{ |z| < 1 \}$ into itself such that $f(0) = f(1/2) = f(-1/2) = 0$. Show that$$\left| f\left( \frac{1}{4} \right) \right| \leq \frac{1}{21}.$$Show that the upper bound $1/21$ cannot be improved. Hint: Use Blaschke products.

Note: I saw problems like this very often on various comprehensive/qualifying exams.

Problem 5. Prove that the sum of the series$$g(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{z + k}$$defines a meromorphic function in $\mathbb{C}$, and identify its poles.

Can a formula for $g'(z)$ be obtained by differentiating the series term-by-term? (give proof either way).
**Problem 6.** Let $U$ be an open set. Suppose the functions $f_n$ are holomorphic in $U$ and converge to $f$ uniformly on any compact subset of $U$; assume that $f$ is not identically zero.

(i) Let $S = \{ z \in U : f_n(z) = 0 \text{ for some } n \}$, i.e. $S$ is the set of all zeroes of all functions $f_n$. Show that the zeroes of $f$ in $U$ are identical with the accumulation points of $S$. More precisely, prove that $a$ is a zero of $f$ if and only if every neighborhood of $a$ contains zeroes of $f_n$ for arbitrarily large $n$.

(ii) Suppose that $U = \mathbb{C}$, so all $f_n$’s are entire functions. Suppose that the functions $f_n$ have only real zeroes. Is it true that $f$ has only real zeroes?

(iii) Suppose $U$ contains the segment $[a, b]$ of the real axis, the functions $f_n$ all assume the real values for real $z$ and have no zeroes on $[a, b]$. Is it true that $f$ has no zeroes on $[a, b]$?