MAT 542 Complex Analysis I

Problem Set 7

due Tuesday, March 27

Problem 1. (i) Let a be real, $0 \le a < 1$. Let U_a be the open set obtained from the unit disk $\{|z| < 1\}$ by removing the segment $[a, 1]$ of the real line. Construct a conformal isomorphism between U_0 and U_a for $a > 0$.

(ii) Construct a conformal isomorphism between U_0 and the unit disk $\{|z| < 1\}$.

Problem 2. Let $f : D \to D$ be a holomorphic map of the unit disk $D = \{|z| < 1\}$ into itself. Show that for all $a \in D$

$$\frac{|f'(a)|}{1 - |f(a)|^2} \le \frac{1}{1 - |a|^2}.$$

Hint: Let g be an automorphism of D that maps 0 to a, and h an automorphism that maps $f(a)$ to 0. Apply the Schwarz lemma to $F = h \circ f \circ g$.

Problem 3. (Blaschke products) A function of the form

$$B(z) = \lambda \left(\frac{z - z_1}{1 - \overline{z_1} z} \right) \cdots \left(\frac{z - z_n}{1 - \overline{z_n} z} \right),$$

where $|\lambda| = 1$ and $|z_j| < 1$ for all $1 \le j \le n$, is called a *finite Blaschke product.*

Check that $B(z)$ is holomorphic in the closed unit disk $\{|z| \le 1\}$, maps this disk into itself and has zeroes at $z_1, \ldots, z_n$.

Suppose that f is holomorphic in the closed unit disk $\{|z| \le 1\}$ and maps this disk into itself. Suppose also that $|f(z)| = 1$ whenever $|z| = 1$. Show that f must be a finite Blaschke product.

Problem 4. Let $f : D \to D$ be a holomorphic map of the unit disk $D = \{|z| < 1\}$ into itself such that $f(0) = f(1/2) = f(-1/2) = 0$. Show that

$$\left| f\left(\frac{1}{4}\right) \right| \le \frac{1}{21}.$$

Show that the upper bound $1/21$ cannot be improved. **Hint:** Use Blaschke products.

Note: I saw problems like this very often on various comprehensive/qualifying exams.

Problem 5. Prove that the sum of the series

$$g(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{z + k}$$

defines a meromorphic function in $\mathbb{C}$, and identify its poles.

Can a formula for $g'(z)$ be obtained by differentiating the series term-by-term? (give proof either way).

Problem 6. Let U be an open set. Suppose the functions f_n are holomorphic in U and converge to f uniformly on any compact subset of U; assume that f is not identically zero.

(i) Let $S = \{z \in U : f_n(z) = 0 \text{ for some } n\}$, i.e S is the set of all zeroes of all functions f_n. Show that the zeroes of f in U are identical with the accumulation points of S. More precisely, prove that a is a zero of f if and only if every neighborhood of a contains zeroes of f_n for arbitrarily large n.

(ii) Suppose that $U = \mathbb{C}$, so all f_n's are entire functions. Suppose that the functions f_n have only real zeroes. Is it true that f has only real zeroes?

(iii) Suppose U contains the segment $[a, b]$ of the real axis, the functions f_n all assume the real values for real z and have no zeroes on $[a, b]$. Is it true that f has no zeroes on $[a, b]$?