HOMEWORK 8 SOLUTIONS

17.4 We wish to prove for any sequence \((x_n) \to x_0\) contained in the domain \([0, \infty)\), that \(\lim \sqrt{x_n} = \sqrt{x_0}\). But this is precisely the statement of example 5 in chapter 8, so we’re done.

17.7
(a) Let \((x_n) \to x_0\) be a convergent sequence. Since constants can be factored through limits, we have \(\lim kx_n = k \lim x_n = kx_0\). This proves \(\lim f(x_n) = f(x_0)\), proving that \(f(x) = kx\) is continuous.

(b) We’ll use the reverse triangle inequality with the \(\epsilon-\delta\) definition. Let \(\epsilon > 0\) be given. By the reverse triangle inequality, \(||x| - |y|| \leq |x - y|\), so we may choose \(\delta = \epsilon\).

(c) It has already been prove that composition of continuous functions is continuous. Therefore if \(f\) is a continuous function, then so are \(kf\) and \(|f|\).

17.9
(a) Given \(\epsilon\), choose \(\delta = \min\{1, \epsilon/5\}\). Then if \(|x - 2| < \delta\), then \(|x - 2| < 1\ so \(|x + 2| < 5\). But we also know that \(|x - 2| < \epsilon/5\), so

\[|x^2 - 4| \leq |x - 2||x + 2| < \epsilon/5 \cdot 5 = \epsilon.\]

(b) Given \(\epsilon\), choose \(\delta = \epsilon^2\). Then if \(|x| = |x - 0| < \delta\), \(\sqrt{|x|} = \sqrt{|x|} < \sqrt{\epsilon} < \epsilon\).

(c) Given \(\epsilon\), choose \(\delta = \epsilon\). Then if \(|x| < \delta\), \(|f(x)| \leq |x| \leq \delta = \epsilon\). Here we used the fact that the \(|\sin(x)| \leq 1\).

(d) First we factor \(|x^3 - x_0^3| \leq |x - x_0||x^2 + xx_0 + x_0^2|\). Now observe that for any \(\delta\), if \(|x - x_0| < \delta\) then \(|x| \leq |x_0| + |x - x_0| < |x_0| + \delta\). Using this in the factorization, we find that if \(|x - x_0| < \delta\) (which is still unspecified),

\[|x^3 - x_0^3| < \delta(|x_0| + \delta)^2 + (|x_0| + \delta)x_0 + x_0^2\]
\[\leq \delta|\delta^2 + 3\delta|x_0| + 3|x_0|^2|.

This shows that given any \(\epsilon\), we can choose \(\delta = \min\{1, \frac{1}{1+3|x_0|+3|x_0|^2}\}\).
17.10

(a) Consider $x_n = 1/n$. Then $x_n \to 0$, but $\lim f(x_n) = \lim 1 = 1 \neq 0 = f(0)$.

(b) Consider $x_n = 1/((\pi/2 + 2n\pi)$. Then $x_n \to 0$, but $\lim g(x_n) = 1 \neq 0 = g(0)$.

(c) Consider $x_n = 1/n$. Then $x_n \to 0$ but $\lim f(x_n) = 1 \neq 0 = f(0)$.

17.12

(a) Given any irrational number $x$, we wish to find a sequence of rational numbers $r_n$ that converges to it. If this can be done, then the continuity condition $f(x) = \lim f(r_n) = 0$ shows the desired result.

For each $n$, choose $r_n$ to be a rational number lying in the interval $(x - 1/n, x + 1/n)$. We know such a number exists by the denseness of rationals, and then $(r_n) \to x$ by construction.

(b) Consider $f - g$. The difference of two continuous functions is continuous, and by assumption this function equals 0 at every rational. By part (a), the difference equals 0 everywhere, which proves that $f = g$ everywhere.