HOMEWORK 8 SOLUTIONS

17.4 We wish to prove for any sequence $(x_n) \to x_0$ contained in the domain $[0, \infty)$, that $\lim \sqrt{x_n} = \sqrt{x_0}$. But this is precisely the statement of example 5 in chapter 8, so we're done.

17.7

(a) Let $(x_n) \to x_0$ be a convergent sequence. Since constants can be factored through limits, we have $\lim kx_n = k \lim x_n = kx_0$. This proves $\lim f(x_n) = f(x_0)$, proving that $f(x) = kx$ is continuous.

(b) We'll use the reverse triangle inequality with the ϵ-δ definition. Let $\epsilon > 0$ be given. By the reverse triangle inequality,

$$||x| - |y|| \leq |x - y|,$$

so we may choose $\delta = \epsilon$.

(c) It has already been prove that composition of continuous functions is continuous. Therefore if f is a continuous function, then so are kf and $|f|$.

17.9

(a) Given ϵ, choose $\delta = \min\{1, \epsilon/5\}$. Then if $|x - 2| < \delta$, then $|x - 2| < 1$ so $|x + 2| < 5$. But we also know that $|x - 2| < \epsilon/5$, so

$$|x^2 - 4| \leq |x - 2||x + 2| < \epsilon/5 \cdot 5 = \epsilon.$$

(b) Given ϵ, choose $\delta = \epsilon^2$. Then if $|x - 0| = |x| < \delta$, $\sqrt{|x|} = \sqrt{x} = |\sqrt{x}| < \epsilon$.

(c) Given ϵ, choose $\delta = \epsilon$. Then if $|x| < \delta$, $|f(x)| \leq |x| < \delta = \epsilon$. Here we used the fact that the $|\sin(x)| \leq 1$.

(d) First we factor $|x^3 - x_0^3| \leq |x - x_0||x^2 + xx_0 + x_0^2|$. Now observe that for any δ, if $|x - x_0| < \delta$ then $|x| \leq |x_0| + |x - x_0| < |x_0| + \delta$. Using this in the factorization, we find that if $|x - x_0| < \delta$ (which is still unspecified),

$$\begin{aligned} |x^3 - x_0^3| &< \delta|(|x_0| + \delta)^2 + (|x_0| + \delta)x_0 + x_0^2| \\ &\leq \delta|\delta^2 + 3\delta|x_0| + 3|x_0|^2|. \end{aligned}$$

This shows that given any ϵ, we can choose $\delta = \min\{1, \frac{1}{1+3|x_0|+3|x_0|^2}\}$.

17.10

(a) Consider $x_n = 1/n$. Then $x_n \to 0$, but $\lim f(x_n) = \lim 1 = 1 \neq 0 = f(0)$.

(b) Consider $x_n = 1/(\pi/2 + 2n\pi)$. Then $x_n \to 0$, but $\lim g(x_n) = 1 \neq 0 = g(0)$.

(c) Consider $x_n = 1/n$. Then $x_n \to 0$ but $\lim f(x_n) = 1 \neq 0 = f(0)$.

17.12

(a) Given any irrational number x, we wish to find a sequence of rational numbers r_n that converges to it. If this can be done, then the continuity condition $f(x) = \lim f(r_n) = 0$ shows the desired result.

For each n, choose r_n to be a rational number lying in the interval $(x - 1/n, x + 1/n)$. We know such a number exists by the denseness of rationals, and then $(r_n) \to x$ by construction.

(b) Consider $f - g$. The difference of two continuous functions is continuous, and by assumption this function equals 0 at every rational. By part (a), the difference equals 0 everywhere, which proves that $f = g$ everywhere.