10.2 Let (s_n) be a bounded decreasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $u = \inf S$. Since S is bounded, u represents a real number. We show $\lim s_n = u$. Let $\epsilon > 0$. Since $u + \epsilon$ is not a lower bound for S, there exists N such that $s_N < u + \epsilon$. Since (s_n) is decreasing, we have $s_n \leq s_N$ for all $n \geq N$. Of course, $s_n \geq u$ for all n, so $n > N$ implies $u \leq s_n < u + \epsilon$, which implies $|s_n - u| < \epsilon$. This shows $\lim s_n = u$.

10.5 Let (s_n) be an unbounded decreasing sequence. Let $M > 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ is unbounded and bounded above by s_1, the set must be unbounded below. Hence for some $N \in \mathbb{N}$ we have $s_N < M$. Clearly $n > N$ implies $s_n \leq s_N < M$, so $\lim s_n = -\infty$.

10.10

(a) This is a straight-foward computation.

$$\begin{aligned} s_2 &= \frac{1}{3}(s_1 + 1) = \frac{1}{3}(1 + 1) = \frac{2}{3}, \\ s_3 &= \frac{1}{3}(s_2 + 1) = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}, \\ s_4 &= \frac{1}{3}(s_3 + 1) = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}. \end{aligned}$$

(b) Part (a) verifies the base case, since all three values are greater than $\frac{1}{2}$. Now assume that $s_n > \frac{1}{2}$ for some n. Then

$$\begin{aligned} s_{n+1} &= \frac{1}{3}(s_n + 1) \\ &> \frac{1}{3}\left(\frac{1}{2} + 1\right) \\ &= \frac{1}{3}\left(\frac{3}{2}\right) \\ &= \frac{1}{2}, \end{aligned}$$

so $s_{n+1} > \frac{1}{2}$. This proves the inductive step. By the principle of mathematical induction, the result is true for all n.

(c) We will use the now-proven fact that $s_n > \frac{1}{2}$.

$$\begin{aligned} s_n &> \tfrac{1}{2} \\ 2s_n &> 1 \\ 3s_n &> s_n + 1 \\ s_n &> \tfrac{1}{3}(s_n + 1) \end{aligned}$$

Note that we began with a known fact, namely that $s_n > \frac{1}{2}$, and then worked towards the desired result. In general, it is not sufficient to work the other way. That is to say, you cannot begin with the desired result and then derive something you know to be true, *unless* each step of the derivation is reversible.

(d) Part (b) shows that (s_n) is bounded below by $\frac{1}{2}$, and part (c) shows that (s_n) is bounded above by $s_1 = 1$. Therefore (s_n) is a bounded sequence. Part (c) also shows that the sequence is monotone, so theorem 10.2 implies that $\lim s_n$ exists.

Since $\lim s_n$ exists, we can define $s = \lim s_n$ as some real number. From the defining equation $s_{n+1} = \frac{1}{3}(s_n + 1)$, we take the limit of both sides to get $s = \frac{1}{3}(s + 1)$. This yields $s = \frac{1}{2}$.

Note that the limit theorems do not explicitly state how to treat something like $\lim(s_n + 1)$. To be fully rigorous in the application of these theorems, you could define the auxiliary sequence (t_n) by $t_n = 1$ for all n. Then $s_n + 1 = s_n + t_n$ for all n, so

$$\lim(s_n + 1) = \lim(s_n + t_n) = \lim s_n + \lim t_n = s + 1.$$

You could also prove the limit directly, and either way is probably overkill.

11.4

(a) w_{2n}, x_{2n}, y_{2n}, z_{8n}.

(b) For w_n, $\pm\infty$. For x_n, 5 and $\frac{1}{5}$. For y_n, 0 and 2. For z_n, 0 and $\pm\infty$.

(d) None of the sequences converge or diverge to $\pm\infty$.

(e) w_n, x_n, y_n are bounded. z_n is unbounded.

11.11 We may assume that $\sup S \notin S$, since otherwise the sequence (s_n) defined by $s_n = \sup S$ trivially satisfies the desired condition.

I will construct a sequence (s_n) and then show $\lim s_n = \sup S$. For notational convenience define $s = \sup S$, and choose $s_1 \in S$ such that $s_1 > s - 1$. This is possible since $s - 1 < s = \sup S$, so $s - 1$ is not an upper bound. Now choose $s_2 \in S$ such that $s_2 > s_1$ and $s_2 > s - \frac{1}{2}$. This is possible by the same reasoning - both s_1 and $s - \frac{1}{2}$ are less than s and therefore cannot be upper bounds. In general, we choose s_{n+1} to satisfy $s_{n+1} > s_n$ and $s_{n+1} > s - \frac{1}{n+1}$.

We now show that $\lim s_n = s$. Let $\epsilon > 0$ be given. Then there exists some N such that $1/N < \epsilon$. For all $n > N$, we have by the above construction $|s - s_n| < 1/n < 1/N < \epsilon$. This proves that $\lim s_n = s$.

Additional problem. Let $\epsilon > 0$ be given. If $\lim a_n = 5$ then there exists a constant N_a such that $n > N_a$ implies $|a_n - 5| < \epsilon/3$. Similarly, since $\lim b_n = 2$, there exists a constant N_b such that $n > N_b$ implies $|b_n - 2| < \epsilon/3$. Now set $N = \max\{N_a, N_b\}$ so that we can control both bounds simultaneously. Then using the triangle inequality,

$$\begin{aligned}
|2a_n - b_n - 8| &= |2a_n - 10 - b_n + 2| \\
&= |2(a_n - 5) + (2 - b_n)| \\
&\leq |a_n - 5| + |a_n - 5| + |2 - b_n| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

This proves that $\lim_{n\to\infty} 2a_n - b_n = 8$.