1. Prove that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ attains its minimum by examining $-f$.

Proof: Since $f$ is continuous on $[a, b]$, so is $-f$. Therefore, by the theorem proved in class, $-f$ attains its maximum value: there exists some $x_0 \in [a, b]$ such that

$$-f(x_0) \geq -f(x) \text{ for all } x \in [a, b].$$

Multiplying this by $-1$, we see that $f(x_0) \leq f(x)$ for all $x \in [a, b]$. Ergo, $f$ achieves a minimum value at $x_0$.

2. Prove that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ attains its minimum by imitating the proof for the maximum:

Proof: The proof that $f$ is bounded on $[a, b]$ remains unchanged, so we know that $m := \inf \{f(x) | a \leq x \leq b\}$ exists. Since $m$ is the greatest lower bound for this set, $m + \frac{1}{n}$ is not a lower bound for any $n$; in other words, for each $n \in \mathbb{N}$, there exists $a \leq x_n \leq b$ such that

$$m \leq f(x_n) < m + \frac{1}{n}.$$

It follows that the sequence of $y$-values $(f(x_n))$ converges to $m$. However, the sequence of $x$-values, $(x_n)$, is only bounded, and not necessarily convergent. But we can use the Bolzano-Weierstrass Theorem to extract a convergent subsequence $(x_{n_k})$. Let us denote the limit of $(x_{n_k})$ by $x_0$. Since each $x_{n_k}$ is between $a$ and $b$, $x_0 \in [a, b]$ as well. Therefore, $f(x_0)$ is defined, and by the continuity of $f$,

$$\lim (f(x_{n_k})) = f(x_0).$$

However, $(f(x_{n_k}))$ is just a subsequence of $(f(x_n))$, and therefore,

$$\lim (f(x_{n_k})) = m.$$

Hence, $f(x_0) = m$, as desired. $f$ achieves a minimum value.

3. (18.5)

(a) Assume $f$ and $g$ are continuous on $[a, b]$, such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that there exists a point $x_0 \in [a, b]$ where $f(x_0) = g(x_0)$.

Proof: Define $h : [a, b] \rightarrow \mathbb{R}$ by $h(x) := f(x) - g(x)$. $h$ is continuous on $[a, b]$. Furthermore,

$$h(a) = f(a) - g(a) \geq 0,$$
$$h(b) = f(b) - g(b) \leq 0.$$

So $h(a) \geq 0 \geq h(b)$, so we can apply the intermediate value theorem to find an $x_0 \in [a, b]$ satisfying $h(x_0) = 0$. Thus, $f(x_0) = g(x_0)$.

(b) Show that Example 1 in the text is a special case of part (a).

We take $f$ defined on $[0, 1]$ as in the example, and $g(x) = x$. The fact that $f(x) \in [0, 1]$ for all $x$ implies, in particular, that

$$f(0) \geq 0 = g(0),$$
$$f(1) \leq 1 = g(1),$$

and we are looking for a point $x_0$ where $f(x_0) = x_0 = g(x_0)$.

4. (18.6) Show that there exists a point $x_0 \in (0, 1)$ where $x_0 2^{x_0} = 1$.

Proof: We will take for granted that $f(x) = x2^x$ is continuous. $f(0) = 0$, and $f(1) = 2$. Since $f(0) < 1 < f(1)$, the intermediate value theorem asserts the existence of a point $x_0 \in (0, 1)$ such that $f(x_0) = 1$. 

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5. (18.7) Suppose that \( f \) is a real-valued continuous function on \( \mathbb{R} \) and that \( f(a)f(b) < 0 \) for some \( a, b \in \mathbb{R} \). Prove that there exists \( x \) between \( a \) and \( b \) such that \( f(x) = 0 \).

Proof: Since \( f(a)f(b) \) is negative, either \( f(a) < 0 < f(b) \) or \( f(b) < 0 < f(a) \). In either case, 0 is an intermediate value, and since \( f \) is continuous, we can apply the intermediate value theorem to find \( x \) between \( a \) and \( b \) such that \( f(x) = 0 \).