MAT 319, Spring 2012
Solutions to HW 9

1. Prove that a continuous function $f : [a,b] \to \mathbb{R}$ attains its minimum by examining $-f$.
Proof: Since f is continuous on $[a,b]$, so is $-f$. Therefore, by the theorem proved in class, $-f$ attains its maximum value: there exists some $x_0 \in [a,b]$ such that

$$-f(x_0) \geq -f(x) \text{ for all } x \in [a,b].$$

Multiplying this by -1, we see that $f(x_0) \leq f(x)$ for all $x \in [a,b]$. Ergo, f achieves a minimum value at x_0.

2. Prove that a continuous function $f : [a,b] \to \mathbb{R}$ attains its minimum by imitating the proof for the maximum:
Proof: The proof that f is bounded on $[a,b]$ remains unchanged, so we know that $m := \inf \{f(x) | a \leq x \leq b\}$ exists. Since m is the greatest lower bound for this set, $m + \frac{1}{n}$ is not a lower bound for any n; in other words, for each $n \in N$, there exists $a \leq x_n \leq b$ such that

$$m \leq f(x_n) < m + \frac{1}{n}.$$

It follows that the sequence of y-values $(f(x_n))$ converges to m. However, the sequence of x-values, (x_n), is only bounded, and not necessarily convergent. But we can use the Bolzano-Weierstrass Theorem to extract a convergent subsequence (x_{n_k}). Let us denote the limit of (x_{n_k}) by x_0. Since each x_{n_k} is between a and b, $x_0 \in [a,b]$ as well. Therefore, $f(x_0)$ is defined, and by the continuity of f,

$$\lim \left(f(x_{n_k})\right) = f\left(x_0\right).$$

However, $\left(f\left(x_{n_k}\right)\right)$ is just a subsequence of $(f(x_n))$, and therefore,

$$\lim \left(f(x_{n_k})\right) = m.$$

Hence, $f(x_0) = m$, as desired. f achieves a minimum value.

3. (18.5)

   (a) Assume f and g are continuous on $[a,b]$, such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that there exists a point $x_0 \in [a,b]$ where $f(x_0) = g(x_0)$.
   Proof: Define $h : [a,b] \to \mathbb{R}$ by $h(x) := f(x) - g(x)$. h is continuous on $[a,b]$. Furthermore,

   $$\begin{aligned} h(a) &= f(a) - g(a) \geq 0, \\ h(b) &= f(b) - g(b) \leq 0. \end{aligned}$$

   So $h(a) \geq 0 \geq h(b)$, so we can apply the intermediate value theorem to find an $x_0 \in [a,b]$ satisfying $h(x_0) = 0$. Thus, $f(x_0) = g(x_0)$.

   (b) Show that Example 1 in the text is a special case of part (a).
   We take f defined on $[0,1]$ as in the example, and $g(x) = x$. The fact that $f(x) \in [0,1]$ for all x implies, in particular, that

   $$\begin{aligned} f(0) &\geq 0 = g(0), \\ f(1) &\leq 1 = g(1), \end{aligned}$$

   and we are looking for a point x_0 where $f(x_0) = x_0 = g(x_0)$.

4. (18.6) Show that there exists a point $x_0 \in (0,1)$ where $x_0 2^{x_0} = 1$.
Proof: We will take for granted that $f(x) = x2^x$ is continuous. $f(0) = 0$, and $f(1) = 2$. Since $f(0) < 1 < f(1)$, the intermediate value theorem asserts the existence of a point $x_0 \in (0,1)$ such that $f(x_0) = 1$.

5. (18.7) Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists x between a and b such that $f(x) = 0$.
Proof: Since $f(a)f(b)$ is negative, either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. In either case, 0 is an intermediate value, and since f is continuous, we can apply the intermediate value theorem to find x between a and b such that $f(x) = 0$.