Problem 1  Let \((X, M, \mu)\) be a measure space. Let \(E_i \in M, i \in \mathbb{N}\) be a collection of sets such that
\[
\sum_{i \geq 0} \mu(E_i) < \infty
\]
Prove that \(\mu(\bigcap_{i \geq 0} E_i) = 0\).

Problem 2  In the assumptions of Problem 1 prove that the set \(A = \{x \in X | x \in E_i\text{ for } \text{infinitely many } i\}\) has measure zero.

Problem 3  Let us call a real number \(x\) trendy if it is the limit of a sequence of rational numbers \(\frac{p_i}{q_i}\) which converge so quickly that
\[
\left| x - \frac{p_i}{q_i} \right| < \frac{1}{q_i^3}
\]
Let us assume that the Borel measure \(\mu\) on \(\mathbb{R}\), which is invariant with respect to translations \(\mu(E + t) = \mu(E), E \in B_\mathbb{R}, t \in \mathbb{R}\) and normalized by \(\mu([0, 1]) = 1\) exists (so called Lebesgue measure). Prove, using results of the previous problems, that the measure of the set \(T\) of trendy numbers is zero. (Hint: Use result of the previous problems to show first that \(\mu(T \cap [0, 1]) = 0\)).

Problem 4  The usual Cantor set is defined by \(C = \bigcap_{i \geq 1} C_i\), where \(C_1\) is the unit interval, each \(C_i\) is the finite union of disjoint closed subintervals of \([0, 1]\) and \(C_{i+1}\) is obtained from \(C_i\) by deleting the middle (open) third of each of the closed subintervals \(C_i\). Show that \(C\) is a Borel measurable subset of \([0, 1]\). Compute the Lebesgue measure of \(C\).

Problem 5  Let \(p(x)\) be a real polynomial. Assume that all solutions to
\[
y' = p(y)
\]
exist for all real \(t\). What is the degree of \(p\)? (Hint analysis of equation \(y' = y^2\) and differential relation \(y' > y^2\) could be a useful intermediate steps of your solution.)
Problem 6  
1. Show that if \( f'' = -f \) on some interval \( I \), then \((f')^2 + f^2\) is constant on \( I \).

2. Consider the differential equation \( g'' = -e^g \) with initial conditions \( g(0) = 0 \) and \( g'(0) = 10 \). Find the maximal interval of existence for \( g \).

Problem 7  
Derive Holder inequality:
\[
\sum_{i=1}^{n} a_i x_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} x_i^q \right)^{\frac{1}{q}}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p \geq 1, q \geq 1, a_i \geq 0, x_i \geq 0 \). (Hint: find the minimum of the function
\[
u = \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} x_i^q \right)^{\frac{1}{q}}
\]
under assumption that \( \sum_{i=1}^{n} a_i x_i = A \) is constant).

Problem 8  
Derive Minkowski inequality:
\[
\left( \sum_{i=1}^{n} (x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( x_i \geq 0, y_i \geq 0 \). (Hint: find the minimum of the function
\[
u = \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}}
\]
under assumption that \( x_i + y_i = z_i \) is constant).

Derive from this inequality
\[
\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}
\]
without restrictions on \( x_i, y_i \).