MAT544 Fall 2009

Homework 9

Problem 1 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E_i \in \mathcal{M}, i \in \mathbb{N}$ be a collection of sets such that

$$\sum_{i\geq 0} \mu(E_i) < \infty$$

Prove that $\mu(\bigcap_{i\geq 0} E_i) = 0$.

Problem 2 In the assumptions of Problem 1 prove that the set $A = \{x \in X | x \in E_i$ for infinitely many $i\}$ has measure zero.

Problem 3 Let us call a real number x *trendy* if it is the limit of a sequence of rational numbers $\frac{p_i}{q_i}$ which converge so quickly that

$$\left| x - \frac{p_i}{q_i} \right| < \frac{1}{q_i^3}$$

Let us assume that the Borel measure μ on $\mathbb{R}$, which is invariant with respect to translations $\mu(E + t) = \mu(E), E \in \mathcal{B}_{\mathbb{R}}, t \in \mathbb{R}$ and normalized by $\mu([0, 1)) = 1$ exists (so called Lebesgue measure). Prove, using results of the previous problems, that the measure of the set T of trendy numbers is zero. (Hint:Use result of the previous problems to show first that $\mu(T \cap [0, 1]) = 0$)

Problem 4 The usual Cantor set is defined by $C = \bigcap_{i\geq 1} C_i$, where C_1 is the unit interval, each C_i is the finite union of disjoint closed subintervals of $[0, 1]$ and C_{i+1} is obtained from C_i by deleting the middle (open) third of each of the closed subintervals C_i. Show that C is a Borel measurable subset of $[0, 1]$. Compute the Lebesgue measure of C.

Problem 5 Let $p(x)$ be a real polynomial. Assume that all solutions to

$$y' = p(y)$$

exist for all real t. What is the degree of p? (Hint analysis of equation $y' = y^2$ and differential relation $y' > y^2$ could be a useful intermediate steps of your solution .)

Problem 6

1. Show that if $f'' = -f$ on some interval I, then $(f')^2 + f^2$ is constant on I.

2. Consider the differential equation $g'' = -e^g$ with initial conditions $g(0) = 0$ and $g'(0) = 10$. Find the maximal interval of existence for g.

Problem 7 Derive Holder inequality:

$$\sum_{i=1}^{n} a_i x_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1, p \ge 1, q \ge 1, a_i \ge 0, x_i \ge 0$. (Hint: find the minimum of the function

$$u = \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}$$

under assumption that $\sum_{i=1}^{n} a_i x_i = A$ is constant).

Problem 8 Derive Minkowski inequality:

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{q} = 1, x_i \ge 0, y_i \ge 0$. (Hint: find the minimum of the function

$$u = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}$$

under assumption that $x_i + y_i = z_i$ is constant).

Derive from this inequality

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

without restrictions on x_i, y_i.