Homework 4: §1.6 - 1*, 3, 5*, 7, 12, 13.
* - in the back of the book

3. Since \( a^{\phi(10)} \equiv 1 \mod 10 \) whenever \( (a, 10) = 1 \), multiplying both sides by \( a \) we get
\[
a^{\phi(10)+1} \equiv a \mod 10, \quad a \in \mathbb{Z}_{10}.
\]
Since \( \phi(10) + 1 = 5 \), we have \( a^5 - a \equiv 0 \mod 10 \).

7. Recall that from Euler’s theorem it follows that \( a^{\phi(n)+1} \equiv a \mod n \) or equivalently \( a^n - a \) is divisible by \( n \) for any \( a \). Thus, \( 2|n^2 - n, 3|n^3 - n, 5|n^5 - n \) and \( 7|n^7 - n \). The result follows from rewriting \( n^{13} - n \) :
\[
\begin{align*}
n^{13} - n &= (n^{11} + n^{10} + \ldots + n + 1)(n^2 - n) \equiv 0 \mod 2 \\
n^{13} - n &= (n^{10} + n^8 + \ldots + n^2 + 1)(n^3 - n) \equiv 0 \mod 3 \\
n^{13} - n &= (n^8 + n^4 + 1)(n^5 - n) \equiv 0 \mod 5 \\
n^{13} - n &= (n^6 + 1)(n^7 - n) \equiv 0 \mod 7.
\end{align*}
\]

12. Since \( 87 = 3 \cdot 29 \) and \( \phi(87) = 56 \), we get \( 3 \cdot 19 - 1 \cdot 56 = 1 \), whence \( x = 3 \). We calculate :
\[
\begin{align*}
4^3 &\equiv 64 \mod 87 \\
10^3 &\equiv 43 \mod 87.
\end{align*}
\]
Thus, the message is FOOD.

13. Since \( 143 = 11 \cdot 13 \) and \( \phi(143) = 120 \), we get \( 7 \cdot 103 - 6 \cdot 120 = 1 \), whence \( x = 7 \). We calculate :
\[
\begin{align*}
10^7 &\equiv 10 \mod 143 \\
3^7 &\equiv 42 \mod 143.
\end{align*}
\]
Thus, the message is JOHN.