Homework 4 : §1.6 - $1^*, 3, 5^*, 7, 12, 13.$
$*$ - in the back of the book

3. Since $a^{\phi(10)} \equiv 1 \bmod 10$ whenever $(a, 10) = 1$, multiplying both sides by a we get

$$a^{\phi(10)+1} \equiv a \bmod 10, \quad a \in \mathbb{Z}_{10}.$$

Since $\phi(10) + 1 = 5$, we have $a^5 - a \equiv 0 \bmod 10$.

7. Recall that from Euler's theorem it follows that $a^{\phi(n)+1} \equiv a \bmod n$ or equivalently $a^n - a$ is divisible by n for any a. Thus, $2|n^2 - n, 3|n^3 - n, 5|n^5 - n$ and $7|n^7 - n$. The result follows from rewriting $n^{13} - n$:

$$\begin{aligned} n^{13} - n &= (n^{11} + n^{10} + \ldots + n + 1)(n^2 - n) \equiv 0 \bmod 2 \\ n^{13} - n &= (n^{10} + n^8 + \ldots + n^2 + 1)(n^3 - n) \equiv 0 \bmod 3 \\ n^{13} - n &= (n^8 + n^4 + 1)(n^5 - n) \equiv 0 \bmod 5 \\ n^{13} - n &= (n^6 + 1)(n^7 - n) \equiv 0 \bmod 7. \end{aligned}$$

12. Since $87 = 3 \cdot 29$ and $\phi(87) = 56$, we get $3 \cdot 19 - 1 \cdot 56 = 1$, whence $x = 3$. We calculate :

$$\begin{aligned} 4^3 &\equiv 64 \bmod 87 \\ 10^3 &\equiv 43 \bmod 87. \end{aligned}$$

Thus, the message is FOOD.

13. Since $143 = 11 \cdot 13$ and $\phi(143) = 120$, we get $7 \cdot 103 - 6 \cdot 120 = 1$, whence $x = 7$. We calculate :

$$\begin{aligned} 10^7 &\equiv 10 \bmod 143 \\ 3^7 &\equiv 42 \bmod 143. \end{aligned}$$

Thus, the message is JOHN.