1. Introduction

There are several numerical functions which can be related in the geometric context of Riemannian manifold, especially those which are complete and have constant negative curvature. They are:

(i) The Hausdorff dimension $D(X)$ of a closed set $X$ in Euclidean space.

(ii) The point $\lambda_0$ of the $L^2$-spectrum nearest to zero for the Friedrich extension of a semidefinite symmetric operator $\Delta$.

(iii) The critical exponent $\delta(\Gamma)$ of the Poincaré-Dirichlet series of a discrete group $\Gamma$ of Möbius transformations of $S^d$.

(iv) The parameter of the spherical complementary series of irreducible representations of $0(n, 1)$, especially $PSL(2, \mathbb{R})$ and $PSL(2, \mathbb{C})$.

(v) The exponential rate of transience of a positivity preserving Markoff operator $P$, equivalently the point nearest to zero in the ‘positive spectrum’ of a Markoff operator $P$, i.e., $\lambda$-potential theory.

(vi) The entropy of an ergodic measure preserving flow.

A rich supply of examples is given by groups $\Gamma$ of non-Euclidean motions of $H^{d+1}$ having finite sided fundamental domains in $H^{d+1}$ whose Poincaré limit set in $S^d$ has Hausdorff dimension $D > \frac{1}{2}d$. Then:

1. The Hausdorff measure $\mu$ of the limit set is finite and positive (when the ranks of any cusps are at most $D$).

2. $\mu$ on the $\lambda$-Martin boundary $S^d$ of $H^{d+1}$ defines a positive $\lambda$-eigenfunction $\phi_\mu$ for the Laplacian which is $\Gamma$-invariant, where $\lambda = D(d - D)$.

In the situation of (1), $\phi_\mu(p)$ has the following nice geometrical interpretation: It is the total Hausdorff mass of the limit set as viewed from $p$.

3. $\phi_\mu$ belongs to $L^2(H^{d+1}/\Gamma)$ and is the lowest eigenfunction or ground state.
(4) $\phi_\mu$ determines an irreducible unitary representation for $0(d + 1, 1)$ with parameter $D$, which is also the critical exponent $\delta(\Gamma)$ of the Poincaré series of $\Gamma$.

(5) $\phi_\mu$ is the unique positive $\lambda$-eigenfunction of $\Delta$, $\Gamma$-invariant, and $\lambda$ is the value nearest to zero. The heat kernel $p_t(x, y)$ on $\mathbb{H}^{d+1}/\Gamma$ decays as $e^{\lambda t}$.

(6) $\mu$ determines an invariant measure for a geodesic flow of total mass $\int q_\mu^2$ whose measure theoretic entropy (and this is also the topological entropy when there are no cusps) is $D$ (see [25], [26], [27], and [28]).

In the first part of the paper we point out the general connections for Riemannian manifolds between (ii) and (v). In the second part we specialize to constant negative curvature and connect up (i), (iii), and (iv) with (ii) and (v). For (vi) see [28]. Most of these results are individually known to separate groups. It is our hope that the entire synthesis may have a certain general interest. (Note, in this the operator $\Delta$ is $\leq 0$.)

2. Statement of results

2.1. Riemannian manifolds: Definition of $\lambda_0(M)$. Let $M$ be an open connected Riemannian manifold without boundary. Define the real number $\lambda_0$ in $(-\infty, 0]$ as the negative of the infimum of $\int_M |\operatorname{grad} \phi|^2 / \int_M |\phi|^2$ over smooth functions $\phi$ on $M$ with compact support. First, we take the $\lambda$-potential theory approach to $\lambda_0(M)$. Say that a smooth function $\phi$ on $M$ is $\lambda$-harmonic if $\Delta \phi = \lambda \phi$, where $\Delta$ is the Laplacian.

**Theorem (2.1).** For each $\lambda \geq \lambda_0$ there are positive $\lambda$-harmonic functions on $M$. For each $\lambda < \lambda_0$ there are no positive $\lambda$-harmonic functions on $M$.

Compare [6], [10], and [19].

Second, we take the Hilbert space approach to $\lambda_0(M)$. There is a canonical self-adjoint operator (also denoted $\Delta$) on $L^2(M)$ extending the Laplacian on smooth function with compact support. If $M$ is complete, all self-adjoint extensions agree and $\Delta$ is unique [12]. In the general case we take for $\Delta$ the infinitesimal generator of the (minimal) heat semigroup, $f(x, t) = \int_M p_t(x, y)f(y)\,dy$. Here the symmetric positive kernel $p_t(x, y)$ is defined to be the supremum (an increasing limit) over all smooth compact subregions with boundary $(M_\alpha, \partial M_\alpha)$ of the fundamental solutions $p_\alpha^*(x, y)$ for the heat equation in $M_\alpha$ vanishing on the boundary $\partial M_\alpha$.

$$p_t(x, y) = \sup_\alpha p_\alpha^*(x, y) \quad \text{and} \quad \left( \frac{\partial}{\partial t} - \Delta_x \right) p_t(x, y) = 0$$

(compare [7]).
Theorem (2.2). The closed $L^2$-spectrum of $\Delta$ contains $\lambda_0$ and is contained in the negative ray $(-\infty, \lambda_0]$.

Compare (Fukushima and Stroock [11]), where the minimal heat kernel $\Delta$ construction above is shown to be equal to the Friedrich abstract minimal self-adjoint extension of the Laplacian.

Corollary (2.3). For $\lambda > \lambda_0$, the symmetric kernel $\int_0^a e^{-\lambda t} p_t(x, y) \, dt$ defines a bounded operator on $L^2$, namely $(\Delta - \lambda)^{-1}$.

Combining Theorems (2.1) and (2.2) we have the following spectral picture for any open Riemannian manifold: $\lambda_0 < 0$ and $\lambda_0$ separates the $L^2$-spectrum from the "positive spectrum": \{ $\lambda \mid \exists \phi < 0, \Delta \phi = \lambda \phi$ \}.

Example (2.4). For $M$ the real line (or Euclidean space) and $\lambda_0 = 0$, the functions $e^{\alpha x}$, $\alpha$ real, are $\alpha^2$-harmonic and \{ $e^{-i\alpha x}$ \} are virtual $L^2$ eigenfunctions belonging to $-\alpha^2$ as continuous spectrum.

Example (2.5). For $M$ the hyperbolic plane and $\lambda_0 = -\frac{1}{2}$, the positive $\lambda$-harmonic functions for $-\frac{1}{2} < \lambda < 0$ are related to the complementary series of $SL(2, \mathbb{R})$ (see §2.3), and the virtual $L^2$ eigenfunctions, as continuous spectrum on $(-\infty, -\frac{1}{2}]$, are related to the principal series of $SL(2, \mathbb{R})$.

Thirdly, we have the Markoff process approach to $\lambda_0$. We say that $\lambda$ belongs to Green's region of $M$ if for some pair $(x, y)$, $x \neq y$.

$$\int_0^a e^{-\lambda t} p_t(x, y) \, dt < \infty.$$ \footnote{Note in this paper the Laplacian is a negative semi-definite operator because of the first figure.}

A variant of a classical proposition (see §5) is that for $\lambda$ in the Green's region the integral converges for all pairs $(x, y)$, $x \neq y$, and defines the $\lambda$-Green's function $g_{\lambda}(x, y)$ which is locally integrable and satisfies

$$(\Delta_x - \lambda)g_{\lambda}(x, y) = \text{dirac mass at } y.$$ So for each $y$, $g_{\lambda}(x, y)$ defines a positive $\lambda$-harmonic function on $M \setminus \{ y \}$.

Theorem (2.6). For any open Riemannian manifold the Green's region consists of either (i) the open ray $(\lambda_0, \infty)$, or (ii) the closed ray $[\lambda_0, \infty)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
In case (i), \(
\int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dt = \infty, \)
\( M \) is said to be \( \lambda_0 \)-recurrent. In case (ii), \(
\int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dt < \infty, \)
\( M \) is said to be \( \lambda_0 \)-transient.

Now we discuss situations in which positive \( \lambda_0 \)-harmonic functions are unique (up to constant multiples).

**Theorem (2.7) (Recurrent case).** If the Green’s region is \((\lambda_0, \infty)\), i.e.,
\(
\int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dt = \infty, \)
then the positive \( \lambda_0 \)-harmonic functions are constant multiples of one another.

**Theorem (2.8) (Square integrable case).** Suppose the spectral measure of \( \Delta \) has an atom at \( \lambda_0 \). Then the \( \lambda_0 \) eigenspace of \( \Delta \) is one-dimensional and is generated by a (square integrable) positive \( \lambda_0 \)-harmonic function \( \phi_0 \).

Also, the integral \(
\int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dy \)
diverges so \( M \) is \( \lambda_0 \)-recurrent and any positive \( \lambda_0 \)-harmonic function is square integrable and thus a multiple of \( \phi_0 \).

We note here the related statement: If any atom of the spectral measure of \( \Delta \) is represented by a (square integrable) positive \( \lambda \)-harmonic function, then \( \lambda = \lambda_0 \) and this atom is situated at \( \lambda_0 \). This follows directly from Theorems (2.1) and (2.2).

**Corollary (2.9).** If a complete manifold \( M \) possess a positive square integrable eigenfunction \( \phi \) for \( \Delta \), then the eigenvalue is \( \lambda_0(M) \) and \( \phi \) is unique up to a constant multiple.

### 2.2. Renormalization of random motion.
Given any positive \( \lambda \)-harmonic function \( \phi \) we can add to the usual random motion on \( M \) a force field or drift term \( \text{grad} \log \phi \). Then we have a biased random motion (the \( \phi \)-process) corresponding to the second order operator \( \Delta + 2 \text{grad} \log \phi \), which acts on functions by (cf. §8)
\[
\Delta f + 2 \text{grad} \log \phi \cdot \text{grad} f.
\]

The transition probabilities for the \( \phi \)-process are
\[
(e^{-\lambda \phi(y)} / \phi(x)) p_t(x, y) \, dy.
\]

When the \( \phi \)-process preserves the constant function 1 we say that \( \phi \) is complete. This amounts to the reproducing formula
\[
\phi(x) = \int_M e^{-\lambda \phi(y)} p_t(x, y) \phi(y) \, dy.
\]
(The inequality \( \geq \) is always true.) When \( \phi \) is complete the \( \phi \)-process also preserves the measure \( \phi^2(y) \, dy \) (cf. §8 and [24]).

When there is only one positive \( \lambda \)-harmonic function up to a multiple we refer to the \( \phi \)-process as the \( \lambda \)-process.

**Theorem (2.10).** Suppose \( M \) is \( \lambda_0 \)-recurrent \((\int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dt = \infty)\). Then the \( \lambda_0 \)-process associated to the second order operator \( \Delta + \text{grad} \log \phi_0 \) preserves the function 1, the measure \( \phi_0^2(y) \, dy \), and is recurrent—almost every
path of the $\lambda_0$-process starting from any point in $M$ enters every set of positive measure infinitely often.

In the square integrable case (Theorem (2.8)) the $\lambda_0$-process preserves a finite measure, $\phi_0^2(y) \, dy$.

2.3. Hyperbolic manifolds. Let $M$ be the unique connected complete simply connected $(d + 1)$-manifold of constant negative curvature $\mathbb{H}^{d+1}$. We recall the two kinds of examples of positive $\lambda$-harmonic functions on $\mathbb{H}^{d+1}$.

First consider a Borel set $A$ in $\mathbb{H}^{d+1}$'s visual sphere at infinity $S^d$ which has finite positive Hausdorff measure in dimension $d$. Define a positive $\alpha(d - d)$-harmonic function $\phi_A$ on $\mathbb{H}^{d+1}$ by the rule: $\phi_A(x) = \text{Hausdorff } \alpha$-measure of $A$ in the visual metric on $S^d$ as viewed from $x$. (That $\phi_A$ is $\lambda$-harmonic follows from the discussion below.)

Second, given $\xi$ in $S^d$ choose stereographic projection of the ball model for $\mathbb{H}^{d+1}$ to the upper half-space model for $\mathbb{H}^{d+1}$ with $\xi \leftrightarrow \infty$. If $y$ is the vertical coordinate, then $\phi(x, a \xi) = (y(x))^\alpha$ is a positive $\alpha(d - d)$-harmonic function on $\mathbb{H}^{d+1}$. (In these coordinates, $\Delta = y^2$ (Euclidean $\Delta) + (1 - d) y \partial / \partial y$.)

Note that in these examples both $\alpha$ and $d - \alpha$ lead to the same eigenvalue $\lambda = \alpha(\alpha - d) = (d - \alpha)((d - \alpha) - d)$. Also $\lambda$ is a minimum $-\frac{1}{4}d^2$ for $\alpha = \frac{1}{2}d$. Theorem (2.11). (i) For $\mathbb{H}^{d+1}$, $\lambda_0 = -\frac{1}{4}d^2$, [17], [21]. (ii) Fix $p \in \mathbb{H}^{d+1}$. Then every positive $\lambda$-harmonic function $\phi$ is uniquely expressible in terms of the $\phi(\cdot, \alpha, \xi)$.

$$\phi(x) = \int_{S^d} \phi(x, \alpha, \xi) \, d\mu(p, \phi)(\xi),$$

where $\alpha = \frac{1}{2}d + (\lambda - \lambda_0)^{1/2}$, the $\phi(\cdot, \alpha, \xi)$ are normalized to be 1 at $p$, and $\mu(p, \phi)$ is a unique positive measure on $S^d$ with total mass $\phi(p)$ [15].

The next two theorems concern the boundary measure $\mu(p, \phi)$ and its measure class for any positive $\lambda$-harmonic function $\phi$. Let $\mu(p, \phi, R)$ be the measure on the sphere $S(p, R)$ of hyperbolic radius $R$ centered at $p$, i.e.,

$$\mu(p, \phi, R) = 1/c_R \cdot (\phi \text{ restricted to } S(p, R)) \cdot \text{spherical measure},$$

where

$$c_R = \begin{cases} e^{-(d-a)R} & \text{if } \lambda > -\frac{1}{4}d^2, \\ Re^{-(d/2)R} & \text{if } \lambda = -\frac{1}{4}d^2, \end{cases}$$

and $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$. 
Theorem (2.12). In the compactified space $H^{d+1} \cup S^d$, the boundary measure $\mu(p, \phi)$ of Theorem (2.11) is constructed from $\phi$ as a weak limit of the $\mu(p, \phi, R)$.

$$\lim_{R \to \infty} \mu(p, \phi, R) = \mu(p, \phi).$$

Now we consider radial limits, along hyperbolic rays $(R, \xi)$ emanating from $p$, of a positive $\lambda$-harmonic function $\phi$ with $\phi(p) = 1$.

Theorem (2.13). (a) For $\xi$ outside the closed support of $\mu(p, \phi)$,

$$\phi(\xi, R) \sim e^{-\alpha R} \text{ as } R \to \infty.$$  

(b) For $\mu(p, \phi)$-almost all $\xi$,

$$\phi(\xi, R) \geq \begin{cases} e^{-(d-\alpha)R} & \text{for } \lambda > -\frac{1}{2}d^2, \\ Re^{(-d/2)R} & \text{for } \lambda = -\frac{1}{2}d^2. \end{cases}$$

(c) For all $\xi$,

$$\phi(\xi, R) \leq e^{\alpha R} \text{ as } R \to \infty.$$  

Again $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{2}d^2)^{1/2}$.

Now we give a generalization of Fatou's theorem. Suppose $\phi_1$ and $\phi_2$ are positive $\lambda$-harmonic functions and $\mu(p, \phi_1)$ is absolutely continuous with respect to $\mu(p, \phi_2)$ with Radon-Nikodým derivative $\psi(\xi)$.

Theorem (2.14). For $\mu(p, \phi_2)$-almost all $\xi$

$$\lim_{R \to \infty} \frac{\phi_1(R, \xi)}{\phi_2(R, \xi)} = \psi(\xi).$$

In particular if $\phi_1 \leq \phi_2$, then $\mu(p, \phi_1) \leq \mu(p, \phi_2)$ by Theorem (2.12), and the conclusion holds.

Define the exponential growth of $\phi$ along a hyperbolic ray $(R, \xi)$ from $p$ in the direction $\xi$ by

$$\limsup_{R \to \infty} \frac{\log \phi(R, \xi)}{R}.$$  

By Theorem (2.13) this growth is always $\leq \alpha = \frac{1}{2}d + (\lambda + \frac{1}{2}d^2)^{1/2}$. Suppose the growth is smaller, $\leq \sigma$, for a set of directions $A \subset S^d$ of positive $\mu(p, \phi)$ measure.

Theorem (2.15). (i) The Hausdorff dimension of $A$ is at least

$$\alpha - \sigma = \left(\frac{1}{2}d + (\lambda + \frac{1}{2}d^2)^{1/2}\right) - \sigma.$$  

(ii) In particular if $\phi$ is bounded, the Hausdorff dimension of any $\mu(p, \phi)$-positive set is at least $\frac{1}{2}d + (\lambda + \frac{1}{2}d^2)^{1/2}$.  

We describe the behavior of the $\lambda$-Green’s function $g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) \, dt$ on $H^{d+1}$, which is finite for $\lambda \in [\lambda_0, \infty)$ and only depends on $r = d(x, y)$ for $r$ near $\infty$. It is convenient to include a description of the $\lambda$-spherical function $S_\lambda(x, y)$ which is by definition the unique (up to a multiple) positive $\lambda$-harmonic function of $x$ in $H^{d+1}$, spherically symmetric about $y$ in $H^{d+1}$. These two functions are solutions of the second order differential equation in the radius $R$ which has regular singular points at $R = 0$ and $R = \infty$.

**Theorem (2.16).** For $\lambda \geq \lambda_0$, $g_\lambda(x, y)$ and $S_\lambda(x, y)$ generate the two-dimensional space of spherically symmetric solutions of $(\Delta - \lambda)f = 0$ on $H^{d+1}\setminus\{y\}$. The $\lambda$-Green’s function ($\int_0^\infty e^{-\lambda t} p_t(x, y) \, dt$) is the small (or recessive) solution near $R = \infty$, and the $\lambda$-spherical function ($\int S^\alpha \phi(x; \xi, \alpha) \, d\theta(\xi)$) is the small (or recessive) solution near $R = 0$.

Thus if $\alpha = \frac{1}{2} d + (\lambda + \frac{1}{4} d^2)^{1/2}$, then $g_\lambda \sim \text{const} \cdot e^{-\alpha R}$ near $R = \infty$, while $S_\lambda \sim \text{const} \cdot e^{-((d-\alpha) R}$ near $R = \infty$, except when $\alpha = \frac{1}{2} d$ where $S_\lambda \sim \text{const} \cdot R e^{-(d/2) R}$ near $R = \infty$.

Now let $\Gamma$ be any discrete group of hyperbolic isometries. If $\Gamma$ has no torsion, then $H^{d+1}/\Gamma$ is a complete Riemannian manifold with constant negative curvature to which the generalities of §2.1 apply. We have the generalized Elstrodt-Patterson theorem.

**Theorem (2.17).** For $M = H^{d+1}/\Gamma$, $\lambda_0(M)$ satisfies

$$
\lambda_0(M) = \begin{cases} 
-\frac{1}{4} d^2, & \text{if } \delta(\Gamma) \leq \frac{1}{2} d, \\
\delta(\Gamma)(\delta(\Gamma) - d), & \text{if } \delta(\Gamma) \geq \frac{1}{2} d,
\end{cases}
$$

where $\delta(\Gamma)$ is the critical exponent of $\Gamma$. (Compare [8], [29].)

Recall the critical exponent $\delta(\Gamma)$ is defined so that the Poincaré series of $\Gamma$,

$$
g(x, y, s) = \sum_{\gamma \in \Gamma} \exp-(sd(x, y))^s,
$$

converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$, where $(x, y)$ is any pair of points in $H^{d+1}$ and $(x, y)$ is the hyperbolic distance.

**Corollary (2.18) (Of proof).** If $M = H^{d+1}/\Gamma$, $M$ is $\lambda_0$-recurrent iff $\delta(\Gamma) \geq \frac{1}{2} d$ and the Poincaré series diverges at $s = \delta(\Gamma)$.

Now $\lambda$-harmonic functions on $M$ are just $\Gamma$-invariant $\lambda$-harmonic functions on $H^{d+1}$. From the definition it follows that for any positive $\lambda$-harmonic function $\phi$ on $H^{d+1}$ and for any isometry $\gamma$ of $H^{d+1}$.

$$
\gamma^*\mu(p, \phi) = |\gamma'|^\alpha \mu(p, \phi \cdot \gamma),
$$

where $|\gamma'|$ is the linear distortion of the visual metric on $S^d$ as viewed from $p$, $\alpha = \frac{1}{2} d + (\lambda + \frac{1}{4} d^2)^{1/2}$ as before, and $\gamma^*\mu(\text{set}) = \mu(\gamma(\text{set}))$. 

Thus if $\phi$ is invariant by $\Gamma$, then $\mu(p,\phi)$ on $S^d$ satisfies

\[(2.1) \quad \gamma^*\mu = |\gamma|^\delta \mu,\]

where $\delta = \alpha$ and $\gamma \in \Gamma$.

Thus Theorem (2.17) yields the existence of measures on $S^d$ satisfying (2.1). Curiously, a bit more can be said about this question than the $\lambda$-potential theory implies. The following theorem generalizes earlier results of Patterson [29] and the author [26].

**Theorem (2.19).** (i) If $\Gamma$ is any discrete group of isometries of $\mathbb{H}^{d+1}$ (except for elementary parabolic or cocompact groups) there is a finite positive measure on $S^d$ satisfying $\gamma^*\mu = |\gamma|^\delta \mu$, $\gamma \in \Gamma$, iff $\delta \in [\delta(\Gamma), \infty)$.

(ii) We may further suppose that $\mu$ is concentrated on the limit set of $\Gamma$ unless $\Gamma$ is geometrically finite without cusps. In these latter cases (including cocompact groups) the only such measure on the limit set is the Hausdorff measure in dimension $\delta(\Gamma)$.

The limit set of $\Gamma$ is by definition the set of cluster points in $S^d$ of any $\Gamma$ orbit in $\mathbb{H}^{d+1}$. The condition geometrically finite without cusps means that $\Gamma$ has a finite sided fundamental domain in $\mathbb{H}^{d+1}$ which does not touch the limit set.

**Remark (2.20).** For the elementary parabolic groups there are point measures in $S^d$ satisfying (2.1) for any $\delta$ in $[0, \infty)$ even though $\delta(\Gamma) = \frac{1}{2}$ (rank of parabolic subgroup) $> 0$.

We mention two more theorems relating the $\lambda$-potential theory of $M = \mathbb{H}^{d+1}/\Gamma$ and the Hausdorff geometry at infinity.

**Theorem (2.21).** (i) If $\Gamma$ is geometrically finite and $M = \mathbb{H}^{d+1}/\Gamma$, then

$$\lambda_0(M) = \begin{cases} -\frac{1}{2}d^2, & D \leq \frac{1}{2}d, \\ D(D - d), & D \geq \frac{1}{2}d, \end{cases}$$

where $D$ is the Hausdorff dimension of the limit set.

(ii) $M$ has a square integrable positive $\lambda_0$-harmonic function iff $D > \frac{1}{2}d$. $M$ is $\lambda_0$-recurrent iff $D \geq \frac{1}{2}d$.

**Corollary (2.22).** Let $M = \mathbb{H}^{d+1}/\Gamma$, where $\Gamma$ is geometrically finite. Then whether or not the Hausdorff dimension of the limit set belongs to $(0, \frac{1}{2}d)$ and if not its exact value in $[\frac{1}{2}d, d]$ is determined by the $\lambda$-potential theory of $M$.

Any discrete group of isometries of the hyperbolic plane $\mathbb{H}^2$ is a union of geometrically finite groups. This allows a general result from [26].

**Theorem (2.23).** For any complete connected hyperbolic surface $S$ let $D$ denote the Hausdorff dimension of the set of those geodesics emanating from any fixed point in $S$ which returns infinitely often to any bounded neighborhood of that
point. Then $\lambda_0(S)$ satisfies

$$
\lambda_0(S) = \begin{cases} 
-\frac{4}{D}, & D \leq \frac{1}{2}, \\
D(D - 1), & D \geq \frac{1}{2}.
\end{cases}
$$

Recall $G(d)$ denotes the group of proper motions of $H^{d+1}$. Then $G(1) = P\ell(2, \mathbb{R})$ and $G(2) = \text{PSL}(2, \mathbb{C})$.

Now Theorem (2.21) allows a canonical geometric interpretation of the complementary series in terms of hyperbolic manifolds $H^{d+1}/\Gamma$ and the Hausdorff geometry of the limit sets of the discrete groups $\Gamma$.

**Theorem (2.24).** Let $\phi_0$ denote the square integrable positive $\lambda_0$-harmonic function on $M = H^{d+1}/\Gamma$, where $\Gamma$ is geometrically finite and the Hausdorff dimension of the limit set $D = \delta(\Gamma) > \frac{1}{2}d$. Then the linear span of the $G(d)$-orbit of $\phi_0$ in $L^2(G(d)/\Gamma)$ generates the member of the complementary series labeled by $\lambda_0(M) \in (-\frac{1}{4}d^2, 0)$.

For example, if $\Gamma$ has no cusps (or all cusps have rank $\leq D$) then $\phi_0(p)$, the $K$-invariant vector, is just the function on $H^{d+1}$ which assigns the Hausdorff $D$-measure of the limit set of $\Gamma$ calculated in the metric as viewed from $p$.

**Remark (2.25).** There are examples where deformations of one $\Gamma$ make $\lambda_0$ cover the entire (spherically symmetric) complementary series [25], [3].

3. Preliminaries: Compact manifolds with smooth boundary

Let $M_a$ be a compact manifold with smooth boundary. Let $p_\alpha(x, y)$ be the fundamental solution of the heat equation in $M_a$ vanishing on $\partial M_a$ (cf. [22]). The infinitesimal generator of the semigroup

$$
f(x, t) = \int_{M_a} p_\alpha(x, y)f(y) \, dy
$$

defines a self-adjoint operator $\Delta$ on $L^2(M_a)$ extending the Laplacian acting on smooth functions vanishing near the boundary [22].

By the compactness of $M_a$ there is a discrete set of eigenvalues for $\Delta$,

$$
\cdots < \lambda_2^a < \lambda_1^a < \lambda_0^a < 0,
$$

and a complete basis of $L^2$ consisting of eigenfunctions vanishing on the boundary.

Since $|\lambda_0^a|$ is the infimum of $\int_{M_a} |\text{grad } \phi|^2 / \int_{M_a} |\phi|^2$ over smooth functions vanishing near the boundary, any eigenfunction $\phi_0$ belonging to $\lambda_0^a$ does not change sign (see §8 for an alternative argument). It follows that $\lambda_0^a$ has multiplicity 1 and $\phi_0$ is unique up to a constant multiple.
Since one may write an absolutely convergent eigenexpansion for \( p_\tau^n(x, y) \),

\[
(3.1) \quad p_\tau^n(x, y) = \sum e^{\lambda_\tau^n \phi_\tau^n(x) \phi_\tau^n(y)}
\]

[22], one has

\[
(3.2) \quad \lim_{\tau \to \infty} e^{-\lambda_\tau^n t} p_\tau^n(x, y) = \phi_\tau^n(x) \phi_\tau^n(y),
\]

where \( \phi_\tau^n \) is the unique positive normalized zeroth eigenfunction.

From the probabilistic interpretation [18] of \( p_\tau^n(x, y) \) dy as the probability density of endpoints of random paths starting at \( x \) which have not hit the boundary before time \( \tau \), one has from (3.2) that the probability of starting from \( x \) and hitting the boundary \( \partial M_\alpha \) by time \( \tau \) is asymptotically 1 like

\[
(3.3) \quad 1 - \text{const} e^{(\lambda_\tau^n)\tau}.
\]

Now recall the Dirichlet problem for \( M_\alpha \). If \( f \) is a continuous function on \( \partial M_\alpha \), then the harmonic extension of \( f \) inside \( M_\alpha \) may be written

\[
(3.4) \quad f(x) = \int_{\partial M_\alpha} f(\xi) \, d\mu_\alpha(x, \xi),
\]

where \( \mu_\alpha(x, \xi) \) is the probability measure associated to hitting the boundary with random paths starting from \( x \).

Now weight the hitting probability by \( e^{-\lambda \tau} \), where \( \tau \) is the hitting time and \( \lambda \) is any number \( > \lambda_\alpha^0 \). By (3.3) the resulting measure \( \mu_\lambda^\alpha(x, \xi) \) is well defined and finite. Again if \( f \) is a continuous function on the boundary, then

\[
(3.5) \quad f(x) = \int_{\partial M_\alpha} f(\xi) \, d\mu_\lambda^\alpha(x, \xi)
\]

defines a smooth \( \lambda \)-harmonic function in \( M_\alpha \) with boundary values \( f \). The classical proof of (3.4) may be modified to give (3.5) replacing \( \Delta \) by \( \Delta - \lambda \).

Now recall that the generalized Poisson measures \( \mu_\alpha(x, \xi) \) of (3.4) are equivalent for various \( x \) and that for fixed \( x_0 \) in \( M_\alpha \) the ratio \( d\mu_\alpha(x, \xi)/d\mu_\alpha(x_0, \xi) = \psi_\alpha(x, \xi) \) is a positive harmonic function for \( \xi \) fixed (which is zero on \( \partial M_\alpha \setminus \{\xi\} \) and has a pole at \( \xi \)). Similarly \( d\mu_\lambda^\alpha(x, \xi)/d\mu_\lambda^\alpha(x_0, \xi) = \psi_\lambda^\alpha(x, \xi) \) is a positive \( \lambda \)-harmonic function on \( M_\alpha \) for \( \xi \) fixed (which is zero on \( \partial M_\alpha \setminus \{\xi\} \) and has a pole at \( \xi \)). (See §2.3 for examples.)
This shows the Harnack principle for positive harmonic functions is also valid for positive \( \lambda \)-harmonic functions, \( \lambda > \lambda_0^\alpha \). Namely, write (3.5) as

\[
(3.6) \quad f(x) = \int_{\partial M_a} f(\xi) \frac{d\mu_\lambda^\alpha(x, \xi)}{d\mu_\lambda^\alpha(x_0, \xi)} d\mu_\lambda^\alpha(x_0, \xi)
\]

\[
= \int_{\partial M_a} f(\xi) \psi_\alpha^\lambda(x, \xi) d\mu_\lambda^\alpha(x_0, \xi),
\]

showing that the values of \( f \) around \( x \) are fixed convex combinations \( (f(\xi) d\mu_\lambda^\alpha(x_0, \xi)) \) of values \( (\psi_\alpha^\lambda(x, \xi)) \) which only vary in a bounded ratio.

### 4. Proof of Theorem (2.1)

Now consider the directed set of all compact connected regions \( M_a \subset M \) with smooth boundary. Since \( \lambda_0^\alpha \) (of §3) is the negative of the infimum over smooth functions supported on interior \( M \) of \( \int_M |\nabla \phi|^2 / \int_M |\phi|^2 \), the number \( \lambda_0 \) defined in the introduction clearly satisfies

\[
\lambda_0 = \sup_\alpha \lambda_0^\alpha,
\]

and \( \lambda_0 > \lambda_0^\alpha \) for all \( \alpha \).

Then by §3 there are positive \( \lambda \)-harmonic functions on \( M_a \) for any \( \lambda > \lambda_0 > \lambda_0^\alpha \). By the Harnack principle described in §3 we have compactness with respect to uniform convergence on compact sets for those positive \( \lambda \)-harmonic functions which are \( < 1 \) at a fixed point \( x_0 \). We can form convergent subsequences of those defined for an exhaustion of \( M \) by \( M_a \) and thereby prove the first part of Theorem (2.1).

The second part of Theorem (2.1) follows from the fact that a positive \( \lambda \)-harmonic function \( f \) continuous on \( M_a \) satisfies

\[
(4.1) \quad f(x) = \int_{M_a} e^{-\lambda t} p_t^\alpha(x, y) f(y) dy + \int_p e^{-\lambda \tau} d(\text{Wiener measure}),
\]

where \( p \) is the set of paths which hit \( \partial M_a \) at \( \tau < t \). So

\[
f(x) > \int_{M_a} e^{-\lambda t} p_t^\alpha(x, y) f(y) dy.
\]

This shows \( \lambda > \lambda_0^\alpha \) using (3.2) and completes the proof of Theorem (2.1).
5. The Green’s region and \( \lambda \)-superharmonic functions

Consider the function \( g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) \, dt \) and suppose \( g_\lambda(x, y) \) is finite for one pair \( x \neq y \). From the definition, \( g_\lambda(x, y) \) is symmetric and as a function of \( x \) it is

1. the increasing limit of continuous functions (and so lower semicontinuous, \( f(x) \leq \lim_{x \to x} f(x_j) \)),

2. decreased pointwise by at least the factor \( e^{\lambda t} \) by the heat semigroup, \( f(x, t) = \int_M p_t(x, y) f(y) \, dy \). Namely, \( f(x, t) \leq e^{-\lambda t} f(x) \).

Functions of \( x \) satisfying (1) and (2) (and not identically \(+ \infty\)) are called \( \lambda \)-superharmonic. So if \( \lambda \) belongs to the Green’s region there is a \( \lambda \)-superharmonic function \( g_\lambda(x, y) \) for each \( y \).

Conversely, suppose \( f \) is \( \lambda \)-superharmonic and let \( P_t^\lambda \) denote \( e^{-\lambda t} \) (heat operator). We apply the operator equation

\[
\int_0^t P_t^\lambda ds \frac{\text{Id} - P_t^\lambda}{t} = \frac{1}{t} \int_0^t P_s^\lambda ds - \frac{1}{t} \int_t^{t+t} P_s^\lambda ds
\]

to \( f \) and deduce using (1) and (2) that either

\[
P_t^\lambda f = f \quad \text{for all } x,
\]
or \( \lambda \) belongs to the Green’s region.

Using the fact that for smooth functions of compact support \( \phi \)

\[
\frac{\text{Id} - P_t^\lambda}{t} \phi \to - (\Delta - \lambda) \phi,
\]
uniformly on compact sets as \( t \to 0 \), one obtains by duality that a \( \lambda \)-superharmonic function (which is locally integrable by \( f \geq P_t^\lambda f \)) satisfies

\[
\lim_{t \to 0} \frac{f - P_t^\lambda f}{t} = - (\Delta - \lambda) f,
\]
in the sense of distributions. Thus \( - (\Delta - \lambda) f \) is a positive Radon measure approximated by \( ((f - P_t^\lambda f) / t) \, dy \), whenever \( f \) is \( \lambda \)-superharmonic.

Calculating the latter for \( g_\lambda(x, y) \) (as a function of \( x \) for \( y \) fixed) yields

\[
\frac{\text{Id} - P_t^\lambda}{t} g_\lambda(x, y) = \frac{1}{t} \int_0^t e^{-\lambda s} p_s(x, y) \, ds
\]
which approaches the Dirac mass at \( y \) as \( t \to 0 \). A corollary is that \( g_\lambda(x, y) \) is finite for all \( x \neq y \) and defines a positive \( \lambda \)-harmonic function on \( M \setminus \{y\} \).

Another corollary is that if \( \lambda \) belongs to the Green’s region, then for every compact \( K \) in \( M \)

\[
\lim_{t \to \infty} e^{-\lambda t} \int_K p_t(x, y) \, dy = 0.
\]
To see this choose $\epsilon_i \to 0$ and $T_i \to \infty$, write
\[
g(x, y) = \lim_{T_i \to \infty, \epsilon_i \to 0} \int_{\epsilon_i}^{T_i} e^{-\lambda t} p_i(x, y) \, dt,
\]
and use the heat equation to calculate $(\Delta_x - \lambda)(g_\lambda(x, y))$. One gets two terms, the one near zero converges to the right answer, the Dirac mass at $y$, so the other one corresponding to $\infty$ must go to zero. Since the convergence is that of radon measures, (5.5) results.

Besides the Green’s function, positive $\lambda$-harmonic functions also provide examples of $\lambda$-superharmonic functions. This follows using (4.1) repeatedly,
\[
p_i(x, y) = \sup_{\alpha} p_{\alpha}^i(x, y) \quad \text{and} \quad M = \bigcup_{\alpha} M_{\alpha}.
\]
More precisely, (4.1) shows that $(\lambda_0, \infty)$ is contained in the Green’s region because the second part (5.2) must hold for a positive $\lambda_0$-harmonic function whenever the $\lambda$ of (4.1) belongs to $(\lambda_0, \infty)$.

Now if $\lambda < \lambda_0$, then $\lambda < \lambda_0^\alpha$ for some $\alpha$ and if $\lambda$ belongs to the Green’s region, (5.5) implies $\int K e^{-\lambda_0 t} p_\alpha(x, y) \, dy \to 0$ as $t \to \infty$ contradicting (3.3). Thus the Green’s region does not contain $\lambda$ and must consist of either $(\lambda_0, \infty)$ or $(\lambda_0, \infty)$. This proves Theorem (2.6).

6. The $L^2$-spectrum of $\Delta$ and the proof of Theorem (2.2)

Using the spectral theorem and the positivity of $p_i(x, y)$ one sees immediately that if the interval $[\lambda, \infty)$ does not intersect the $L^2$-spectrum of $\Delta$ (the infinitesimal generator of the semigroup $f(x, t) = \int_M p_i(x, y) f(y) \, dy$), then the bounded operator on $L^2$, $(\Delta - \lambda)^{-1}$, is represented by the positive kernel $\int_0^\infty e^{-\lambda t} p_i(x, y) \, dy$. Applying the operator to a positive function with compact support shows that $\int_0^\infty e^{-\lambda t} p_i(x, y) \, dy$ is finite a.e. Thus $[\lambda, \infty)$ is contained in the Green’s region. So the entire component of the complement of the spectrum containing the positive reals is contained in the Green’s region.

For the other inequality required by Theorem (2.2) consider the $L^2$-norm of $P_t f = \int_M p_i(x, y) f(y) \, dy$. This is the square root of
\[
\int_M \left( \int p_i(x, y_1) f(y_1) \, dy_1 \int p_i(x, y_2) f(y_2) \, dy_2 \right) \, dx.
\]
Thus,
\[
\| P_t f \|_{L^2} = \left( \int_{M \times M} p_{2i}(y_1, y_2) f(y_1) f(y_2) \, dy_1 \, dy_2 \right)^{1/2},
\]
by the semigroup equation for $p_i(x, y)$. 

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Now consider a positive, bounded, measurable \( f \), with support contained in a compact \( K \) in the interior of \( M \). By (5.5) for each \( y_2 \), 
\[
e^{-\lambda t} p_t(x, y_1) f(y_1) \, dy_1 \to 0 \quad \text{as} \quad t \to \infty \text{ if } \lambda \text{ belongs to the Green’s region.}
\]
For a set \( A \) of \( y_2 \)'s in \( K \) of almost full measure this convergence is uniform. Thus if \( g \) is \( f \) times the characteristic function of \( A \), we have, by (6.1), that the \( L^2 \)-norm of \( P_t g \) times \( e^{-\lambda t} \) goes to zero as \( t \to \infty \). The linear span of these \( g \) is dense in \( L^2 \). It follows the \( L^2 \)-spectrum of \( \Delta \) cannot have points greater than \( \lambda \), for then there would be elements \( h \) in \( L^2 \) so that the \( L^2 \)-norm of \( P_t h \) would not decrease as fast as \( e^{\lambda t} \). This proves Theorem (2.2).

The corollary to Theorem (2.2) is explained by the first paragraph of this section.

7. On the uniqueness of positive \( \lambda_0 \)-harmonic functions

(Proofs of Theorems (2.7) and (2.8))

Suppose the convex cone of positive \( \lambda_0 \)-harmonic functions is not a single ray. The base of this cone \( \{ \phi \mid \phi(x_0) = 1 \} \) is convex, metrizable, and compact in the topology of uniform convergence on compact sets by the Harnack principle of §3. Let \( f \) and \( g \) be two different extreme points of this compact convex set so that \( f \leq g \) and \( g \leq f \) are both false and form \( \phi = \min\{ f, g \} \).

Let \( P_t^{\lambda_0} \) be the operator of §5. From (4.8) it follows that \( P_t^{\lambda_0} f \leq f \) and \( P_t^{\lambda_0} g \leq g \). Thus by positivity of \( P_t^{\lambda_0} \), \( P_t^{\lambda_0} \phi \leq \phi \) so \( \phi \) is \( \lambda_0 \)-superharmonic (§5). Since \( \phi \) is not smooth \( \phi \) cannot be \( \lambda_0 \)-harmonic. (There is a transversality detail here which can be treated using multiples of \( f \) and \( g \) if necessary.) Thus, \( P_t^{\lambda_0} \phi \neq \phi \) for some \( t \) and the second case of (5.2) must hold. Thus \( \lambda_0 \) belongs to the Green’s region, i.e. \( \int_0^\infty e^{-\lambda_0 t} p_t(x, y) \, dt < \infty \). This proves Theorem (2.7).

Now suppose there is an atom at \( \lambda_0 \) for the spectral measure of \( \Delta \) on \( L^2 \). Since \( \Delta - \lambda_0 \) is the infinitesimal generator of \( P_t^{-\lambda_0} \) we must have \( P_t^{-\lambda_0} \phi = \phi \) for \( \phi \) in the \( \lambda_0 \) eigenspace of \( \Delta \). In particular, \( \|P_t^{-\lambda_0} g\| \) does not approach zero as \( t \to \infty \) for a dense set of \( L^2 \). Thus by (5.5) \( \lambda_0 \) is not in the Green’s region. This proves the second part of Theorem (2.8).

Now we give a proof that any \( \phi \) in \( L^2 \) satisfying \( P_t^{\lambda_0} \phi = \phi \) cannot change sign. By Theorem (2.2), \( P_t^{\lambda_0} \) is a contraction on \( L^2 \), so \( \|P_t^{\lambda_0} \phi\|_2 \leq \|\phi\|_2 \) where \( |\phi| \) is the absolute value of \( \phi \). On the other hand,
\[
|\phi(x)| = |P_t^{\lambda_0} \phi(x)| \leq P_t^{\lambda_0} |\phi|(x),
\]
so \((|\phi(x)|)^2 \leq (P_t^{\lambda_0} |\phi(x)|)^2\). Combining these two gives \( |\phi(x)| = P_t^{\lambda_0} |\phi(x)| \) a.e.
If $\phi$ is not entirely negative, at a generic point where $\phi(x) > 0$ we have

$$\phi(x) = \int_M e^{-\lambda_0 t} p_t(x, y) \phi(y) \, dy,$$

so $\phi = |\phi|$ a.e. and $\phi$ must be entirely positive.

Since any $\phi$ does not change sign no two can be orthogonal in $L^2$. This completes the proof of Theorem (2.8).

8. The $\phi$-process and completeness of $\lambda$-harmonic functions

(Proof of Theorem (2.10))

It is formal that the operator defined on functions by the kernel $e^{-\lambda t} \phi(y)/\phi(x) p_t(x, y)$ and on measures by duality preserves the function 1 and the measure $\phi^2(y) \, dy$ iff $\phi(x) = \int_M e^{-\lambda t} p_t(x, y) \phi(y) \, dy$ (i.e. $\phi$ is complete in the terminology of §2.2).

The differential operator or infinitesimal generator associated to this diffusion operator is $[\phi]^{-1} (\Delta - \lambda)[\phi]$, where $[\phi]$ denotes the multiplication operator by $\phi$. Thus

$$[\phi]^{-1} (\Delta - \lambda)[\phi] f = \phi^{-1} (\Delta - \lambda) \phi f$$
$$= \phi^{-1} ((\Delta \phi) \cdot f + \phi \cdot \Delta f + 2 \, \text{grad} \, \phi \cdot \text{grad} \, f - \lambda \phi f)$$
$$= \Delta f + 2 \, \text{grad} \, \log \phi \cdot \text{grad} \, f,$$

since $\Delta \phi = \lambda \phi$.

If $M$ is $\lambda_0$-recurrent and $\phi_0$ is the unique positive $\lambda_0$-harmonic function (up to a multiple), then by (5.2) we must have $\phi_0(x) = \int_M e^{-\lambda_0 t} p_t(x, y) \phi_0(y) \, dy$, namely the first of (5.2) holds. For otherwise, by the second of (5.2), $\lambda_0$ belongs to the Green's region. This proves all but the last part of Theorem (2.10).

To prove recurrence we simply check the criterion for recurrence that the Green's function of the process is identically $+\infty$. For the $\phi$-process the Green's function is $\int_0^\infty e^{-\lambda_0 t} \phi(y)/\phi(x) p_t(x, y) \, dt$ which equals $+\infty$ since the $\phi(y)/\phi(x)$ factor does not matter. This proves Theorem (2.10).

Now let us discuss the question of completeness for $\lambda$-harmonic functions. We will give several arguments for the existence of complete $\lambda$-harmonic functions which depend on auxiliary hypotheses.

Argument (8.1) (Fixed point property). Let $\mathcal{C}_\lambda$ note the convex cone of positive $\lambda$-superharmonic functions. The heat semigroup operates on $\mathcal{C}_\lambda$. Using compactness of the base of $\mathcal{C}_\lambda$ and continuity of $P_t$ (if true simultaneously) we
have, by the fixed point theorem, fixed rays in $\mathcal{C}_\lambda$. Taking the minimum $\lambda$, namely $\lambda_0$, the equation $P_t \phi = c \phi$ implies $c = e^{-\lambda_0 t}$ and we arrive at a complete positive $\lambda_0$-harmonic function. (I am indebted to Dan Stroock for pointing out that a topology making $\mathcal{K}$ have a compact base and $P_t$ continuous for a general Riemannian manifold is not obvious.)

**Argument (8.2) (Minimal $\lambda$-harmonic functions).** Let $\mathcal{K}_\lambda$ denote the convex cone of nonnegative $\lambda$-harmonic functions. The base of $\mathcal{K}_\lambda$ is compact by the Harnack principle of §3. Suppose the heat semigroup preserves $\mathcal{K}_\lambda$ or that even $\mathcal{K} = \mathcal{K}_\lambda \cap P_t \mathcal{K}_\lambda \neq 0$ is a nontrivial convex cone with a compact base. Let $f$ lie in an extreme ray of $\mathcal{K}$ and let $f^0 = P_t^\lambda f$. Then $f \geq f^0$ by (4.1) and $f^0$ belongs to $\mathcal{K}$. Now $g = f - f^0$ is nonnegative and $\lambda$-harmonic. If $f = P_t^\lambda h$, then $g = P_t^\lambda (h - f)$ so $g$ belongs to $\mathcal{K}$. Since $f = g + f^0$ we must have $g = c_1 f$ and $f^0 = c_2 f$ since $f$ is extreme. But $c_2 < 1$ is impossible for then $f^0$ would not be $\lambda$-harmonic. Thus $f = P_t^\lambda f$ for any extreme ray. By linearity and Choquet, $h = P_t^\lambda h$ for any $h$ in $\mathcal{K}$.

So if $\mathcal{K} = \mathcal{K}_\lambda \cap P_t \mathcal{K}_\lambda$ is closed and nontrivial it consists entirely of complete $\lambda$-harmonic functions.

**Example (8.3).** If $M$ is the interior of a compact manifold with boundary, a continuous positive $\lambda$-harmonic function $\phi$ is rarely complete. By (4.1) it is necessary that $\phi$ vanishes on the boundary. Thus $\lambda = \lambda_0$ and $\phi$ must be proportional to zeroth eigenfunction $\phi_0$, which is complete.

**Example (8.4) (Another argument).** If $M$ (or a covering space) has bounded geometry, that is each point is centered in a neighborhood of fixed radius which is a bounded distortion of the unit ball in Euclidean space, then every positive $\lambda$-harmonic function is complete. This follows because the constants in Harnack’s principle are uniform (so a positive $\lambda$-harmonic function $\phi$ grows at most exponentially) and the heat kernel satisfies an inequality $p_t(x, y) \leq ce^{a(d(x, y))\lambda}$ for $t \leq 1$ and $d(x, y) \geq 1$ (so $p_t(x, y)\phi(y) dy$ has little mass near infinity). Now a straightforward estimate shows that a positive $\lambda$-harmonic function is complete.

**Problem (8.5) (Stroock and Sullivan).** Which open connected manifolds have complete positive $\lambda_0$-harmonic functions? (See [24].)

We now turn to the proofs of the theorems in §2.3.

### 9. Proof of Theorems (2.11) and (2.16)

If for some $\lambda$ there is a positive $\lambda$-harmonic function $\phi$ on $H^{d+1}$, then we can average $\phi$ over the compact group of isometries fixing some $y$ in $H^{d+1}$. We obtain a spherically symmetric positive $\lambda$-harmonic function
\[ \phi(R) = S_\lambda(x, y), \] where \( R = d(x, y). \) Then \( \phi(R) \) satisfies

\[ \left( \frac{d^2}{dR^2} + \frac{A'(R)}{A(R)} \frac{d}{dR} - \lambda \right) \phi = 0, \]

where \( A(R) = \) the area of the sphere of radius \( R \) about \( y, \) and \( A'(R) = (d/dR)A(R). \)

For \( R \) near zero and infinity respectively, this equation becomes

\[ R = 0: \quad \left( \frac{d^2}{dR^2} + \frac{d + 1}{R} \frac{d}{dR} - \lambda \right) \phi = 0, \]

\[ R = \infty: \quad \left( \frac{d^2}{dR^2} + d \frac{d}{dR} - \lambda \right) \phi = 0. \]

The exponential solutions near \( \infty \) are determined from the indicial equation

\[ u^2 + du - \lambda = 0. \] In other words if \( \alpha = -u, \lambda = \alpha(\alpha - d). \) Real exponentials result iff \( \lambda \geq -\frac{1}{4}d^2. \) Thus there are spherically symmetric positive \( \lambda \)-harmonic functions iff \( \lambda \geq -\frac{1}{4}d^2. \) This proves Theorem (2.11)(i).

Before proving Theorem (2.11)(ii) we must prove Theorem (2.16) and analyze the \( \lambda \)-Green's function,

\[ g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_\lambda(x, y) \, dt. \] Looking again at the equations in the form (9.2) one sees:

(i) Near \( R = 0 \) there is a 1-dimensional subspace of bounded solutions, the rest of the solutions have a standard Green's singularity, \( \log(1/R) \) if \( d = 1 \) and \( (1/R)^{d-1} \) if \( d > 1. \)

(ii) At \( R = \infty \) there is a 1-dimensional space of solutions asymptotic to \( \text{const} \cdot e^{-\alpha R}, \) where \( \alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}. \) The rest are asymptotic to \( \text{const} \cdot e^{-\left((d-\alpha)R\right)} \) if \( \alpha > \frac{1}{2}d \) or \( \text{const} \cdot Re^{-\left(d/2\right)R} \) if \( \alpha = \frac{1}{2}d. \)

We know from Theorem (2.11) and Theorem (2.6) and the nonuniqueness of positive \( \lambda_0 \)-harmonic functions that the Green's region is \( [\lambda_0, \infty). \) We know from \( (\Delta - \lambda)g_\lambda(x, y) = \) dirac mass at \( y \) that \( g_\lambda(x, y) \) has a standard Green's singularity at \( x = y, \ R = 0. \)

We have seen from the definition that \( S_\lambda(x, y) \) is bounded near \( R = 0 \) and therefore \( S_\lambda(x, y) \) is the small (or recessive) solution near \( R = 0. \) We want to show that \( g_\lambda(x, y) \) is the small (or recessive) solution at \( R = \infty. \)

**Claim (9.1).** The recessive solution at \( R = \infty \) for \( \lambda \geq \lambda_0 \) is positive for all \( R > 0 \) and has a Green's singularity at \( R = 0. \)

**Proof of claim.** The bounded solution at \( R = 0, S_\lambda(x, y), \) has the simple formula

\[ \int \phi(x; \xi, \alpha) \, d\theta, \]
where \( d\theta \) is the spherical measure on \( S^d \) with \( y \) the center of the unit ball model and the \( \phi(\cdot; \xi, \alpha) \) of §2.3 are normalized at \( y \).

A special case of the calculation in the proposition of the proof of Theorem (2.13) shows that \( S_\lambda \) is a large solution near \( R = \infty \). Thus \( g \), the recessive solution at \( R = \infty \), cannot also be recessive at \( R = 0 \) because it would then be a multiple of \( S_\lambda \) (which is large at \( R = \infty \)).

Thus \( g \) tends to \( \infty \) as \( R \to 0 \) and must cross \( S_\lambda \) for some smallest \( R = R_0 \). At \( R_0 \) the Wronskian \( gS'_\lambda - S_\lambda g' = g(R_0)(S'_\lambda - g') \) is negative since \( g(R_0) = S_\lambda(R_0) > 0 \), and \( S'_\lambda(R_0) < g'(R_0) \). Since the Wronskian does not change sign and \( S'_\lambda < 0 \), each of the behaviors depicted in Figure 2 is ruled out. So \( g > 0 \) and we have Figure 3 which proves the claim and a bit more. q.e.d.

To finish the proof that \( g = \text{const} \cdot g_\lambda(x, y) \) write \( g_\lambda(x, y) \) as the \( \sup_\alpha g^\alpha_\lambda(x, y) \), where \( D_\alpha \) is an exhaustion of \( H^{d+1} \) by balls centered at \( y \), and
$g(x, y)$ is the $\lambda$-Green's function for $D_\alpha$. Now $c_1 g - c_2 S_\lambda$ is zero on $\partial D_\alpha$ and has the same weight singularity at $R = 0$, where $c_1$ and $c_2$ are positive constants. So $c_1 g - c_2 S_\lambda$ is zero on $dD_\alpha$ and has the same weight singularity at $R = 0$, where $c_1 x$ and $c_2$ are positive constants. So $c_1 g - c_2 S_\lambda = g\chi(x, y)$. Thus $g_\lambda(x, y) \leq \text{const} \cdot g$. The constant is fixed, so $g_\lambda(x, y) = \sup_\alpha g_\lambda(x, y) \leq \text{const} \cdot g$. It follows that $g_\lambda(x, y)$ is small (or recessive) at $R = \infty$ and must be a constant times $g$. This completes the proof of Theorem (2.16).

Now we are in a position to prove Theorem (2.11)(ii) by Martin's construction (1941). We sketch the steps of this famous argument.

Choose a reference point $x_0$ in $H^{d+1}$ and consider the quotient $k_\lambda(x, y) = g_\lambda(x, y)/g\chi(x_0, y)$. As a function of $y$ ($x$ fixed), $k_\lambda(x, y)$ is continuous on $H^{d+1} \cup S^d$ with $k_\lambda(x, \xi) = \phi(x, \xi, \alpha)$ (normalized at $x_0$) for $\xi \in S^d$. This follows from Theorem (2.16), $\alpha = \frac{1}{2}(\Delta + \frac{1}{2}d^2)^{1/2}$.

Let $\phi$ be a positive $\lambda$-harmonic function which is a limit of $\lambda$-potentials

$$\phi_n(x) = \int_y g_\lambda(x, y) \, d\mu_n(y)$$

of Radon-measures $\mu_n$ on $H^{d+1}$ (all are as we shall see). The measures $\mu'_n = g_\lambda(x_0, y)\mu_n$ have total mass $\leq \phi_n(x)$ ($\leq \phi(x) + 1$ for $n$ large). So let $\mu$ be a weak limit measure in $H^{d+1} \cup S^d$. Since $(\Delta - \lambda)\phi_n = \mu_n$ and $(\Delta - \lambda)\phi = 0$, $\mu$ must be supported on $S^d$. We calculate

$$\phi(x) = \lim_n \phi_n(x) = \lim_n \int_y g_\lambda(x, y) \, d\mu_n(y) = \lim_n \int_y k_\lambda(x, y) \, d\mu'_n(y)$$

$$= \int_y k_\lambda(x, y) \, d\mu \quad \text{(because $k_\lambda(x, y)$ is a continuous function of $y$)}$$

$$= \int_\xi \phi(x, \xi, \alpha) \, d\mu(\xi),$$

since $\mu$ lives on $S^d$. This proves the existence part of Theorem (2.11)(ii) for a limit of potentials.

We now give the classical argument to see that any $\lambda$-superharmonic function $f$ is an increasing limit of potentials. Form $f_n = \min\{f, nG_\lambda\chi_n\}$, where $\chi_n$ is the characteristic function of the ball of radius $n$ about some fixed point and $G_\lambda\chi_n(x) = \int_{B^n} \chi_n(y) \, g_\lambda(x, y) \, dy$. Then $f_n$ is nonnegative bounded, $\lambda$-superharmonic, $f_n$ increases to $f$, and $f_n$ satisfies $\inf_{T \to \infty} P^\lambda_{\chi_n} f_n = 0$ (the latter, since this is true for $nG_\lambda\chi_n$ and $\inf\{P^\lambda_{\chi_n} f, P^\lambda_{\chi_n} g\} \geq P^\lambda_T \inf\{f, g\}$).

Now apply (5.1) to $f_n$ and let $T \to \infty$ to obtain

$$G_\lambda(1/t(f_n - P^\lambda_{\chi_n} f_n)) = 1/t \int_0^t P^\lambda_T f_n \, ds.$$
The right-hand side is increasing to \( f_n \) as \( t \to 0 \) since \( f_n \) is \( \lambda \)-superharmonic. Thus \( f_n \) is the increasing limit of potentials \( G_{\lambda \mu} \), where \( \mu_r = 1/t(f_n - P_{\lambda}f_n) \). This implies that \( f \) is the increasing union of potentials and completes the proof of the existence part of Theorem (2.11)(ii).

The uniqueness follows from Theorem (2.12) (which only uses the existence part of Theorem (2.11)(ii) in its proof).

10. Proof of Theorems (2.12), (2.13), (2.14), and (2.15)

To prove Theorem (2.12) we must first calculate the normalizing factor for \( \mu(p, \phi, R) = 1/c_R \cdot (\phi/S(p, R)) \cdot \text{spherical measure} \). We want

\[
\phi(p) = 1/c_R \int \phi/S(p, R) d\theta_R(x),
\]

where \( d\theta_R \) is the unit spherical measure on \( S(p, R) \). Write \( \phi \) as an integral of the \( \phi(\cdot, \xi, \alpha) \),

\[
\phi(x) = \int_{\xi} \phi(x, \xi, \alpha) d\mu(p, \phi)(\xi),
\]

where \( \mu(p, \phi) \) has total mass \( \phi(p) \). Substituting gives

\[
\phi(p)c_R = \int_{x} \int_{\xi} \phi(x, \xi, \alpha) d\mu(p, \phi)(\xi) d\theta_R(x)
= \int_{\xi} \left( \int_{x} \phi(x, \xi, \alpha) d\theta_R(x) \right) d\mu(p, \phi)(\xi).
\]

Thus \( c_R \) is the function of \( R, S_\lambda(R) = \int x \phi(x, \xi, \alpha) d\theta_R(x) \) where \( x = (R, \xi) \), which we have seen in §9 to be of the order \( e^{-(d-a)R} \) for \( \alpha > \frac{1}{2}d \) and \( Re^{-(d/2)R} \) for \( \alpha = \frac{1}{2}d \). With the indicated choice of \( c_R \) the total mass of \( \mu(p, \phi, R) \) is \( \phi(p) \).

Now let \( f \) be a continuous function on the compactification of \( H^{d+1} \) by \( H^{d+1} \cup S^d \) and let \( R \to \infty \). Then

\[
\frac{1}{c_R} \int f d\mu(p, \phi, R) = \frac{1}{c_R} \int f \cdot \phi \cdot d\theta_R
= \frac{1}{c_R} \int f \left( \int_{\xi} \phi(x, \xi, \alpha) d\mu(p, \phi)(\xi) \right) d\theta_R(x)
= \int_{\xi} \left( \frac{1}{c_R} \int f(x) \cdot \phi(x, \xi, \alpha) d\theta_R(x) \right) d\mu(p, \phi)(\xi).
\]
Outside a disk of radius $\varepsilon > 0$ (fixed so that $f$ is near $f(\xi)$ on this region) in polar coordinates $(R, \xi)$, $\phi(x, \xi, \alpha)$ is of the order $e^{-\alpha R}$. On the other hand, the integral $\int \phi(x, \xi, \alpha) d\theta R$ is larger, $e^{-(d-\alpha)R}$ or $Re^{-(d/2)R}$ as indicated above.

Thus the inner integral is concentrated near $\xi$ and converges to $f(\xi)$ on $R \to \infty$. Thus

$$\lim_{R \to \infty} \int f \cdot d\mu(p, \phi, R) = \int f d\mu(p, \phi),$$

proving Theorem (2.12).

**Remark (10.1).** This proof of Theorem (2.12) for $\alpha > \frac{1}{2}d$ was shown to me by Mary Rees who offered it as an alternative to the sketch of part (ii) for $\alpha > \frac{1}{2}d$ in [26]. The questions of Mary Rees were part of the motivation for the exposition here.

Now we prove Theorem (2.13). First we have a proposition asserting that no finite measure $\mu$ on $S^d$ is more diffuse than Lebesgue measure.

**Proposition (10.2).** Let $\mu$ be a finite positive measure on $S^d$. Then for $\mu$-almost all $\xi$ in $S^d$,

$$\liminf_{r \to 0} \frac{\mu(\xi, r)}{r^d} > 0,$$

where $\mu(\xi, r)$ is the $\mu$-measure of a disk of radius $r$ centered at $\xi$.

**Proof.** Let $A$ be the set of $\xi$ in $S^d$ so that for every $\delta > 0$ and $\xi$ in $A$ there is a sequence $r_i \to 0$ with $\mu(\xi, r_i) \leq \delta r_i^d$. By the covering lemma [9, Theorem 2.8.14] there are (arbitrarily fine) coverings of $A$ using disks of these radii (and centers on $A$) which fall into $K = K(d)$ collections consisting of disjoint disks.

One of these collections $C$ must contain at least $1/K \cdot \mu(A)$ of the mass of $\mu$. Thus

$$1/K \cdot \mu(A) \leq \sum_{C} \mu(\xi, r_i) \leq \varepsilon \sum r_i^d$$

$$\leq \varepsilon \cdot \text{Lebesgue measure of } S^d.$$

So $\mu(A) \leq \varepsilon \cdot K \cdot \text{measure of } S^d$ for any $\varepsilon > 0$. This proves the proposition.

Fix $\xi_0$ and calculate for $x = (R, \xi_0)$

$$\phi(x) = \int_{\xi} \phi(x, \xi, \alpha) d\mu(p, \phi)(\xi).$$

Divide the integral into three parts: (i) $d(\xi, \xi_0) \leq e^{-R}$, (ii) $e^{-R} \leq d(\xi, \xi_0) \leq \varepsilon$, and (iii) $d(\xi, \xi_0) \geq \varepsilon$. Here $\varepsilon > 0$ is a parameter and $d$ is the spherical or Euclidean distance in the unit ball model.
An elementary calculation (see [27, §1]) shows that for \( x = (\xi_0, R) \) in these three regions \( \phi(x, \xi, a) \) is comparable to

(i) \( e^{+\alpha R} \),  
(ii) \( e^{-\alpha R/s^2} \),  
(iii) \( e^{-\alpha R} \),

where \( s = d(\xi, \xi_0) \). Thus

\[
\phi(x) = \phi(\xi, R) = \int_{(i)} e^{\alpha R} d\mu + \int_{(ii)} e^{-\alpha R/s^2} d\mu + \int_{(iii)} e^{-\alpha R} d\mu.
\]

The first term is comparable to \( e^{\alpha R} \mu(\xi_0, e^{-\alpha R}) \). The third term is at most \( e^{-\alpha R} \). We treat the second term by partial integration to obtain (ignoring constants)

\[
e^{-\alpha R} \left( \int_{e^{-\alpha R} < s < e^{\alpha R}} \mu(\xi_0, s)/s^{2a+1} \, ds - \mu(\xi_0, e^{-\alpha R})/e^{-\alpha R} + C(\varepsilon) \right)
\]

\[
= e^{-\alpha R} \int_{e^{-\alpha R} < s < e^{\alpha R}} \mu(\xi_0, s)/s^{2a+1} \, ds - (\text{const} \cdot \text{first term}) + \text{const}(\varepsilon).
\]

Now by the previous proposition for \( \mu \)-almost all \( \xi_0, \mu(\xi_0, s) \) is eventually \( \geq c(\xi_0)s^d \). So, II = \( e^{-\alpha R/2} \int_{\xi} \mu(\xi_0, s)/s^{2a+1} \, ds \) is

\[
\geq \begin{cases}  
  e^{-(d-\alpha)R} & \text{if } \alpha > \frac{1}{2}d, \\
  Re^{-(d/2)R} & \text{if } \alpha = \frac{1}{2}d.
\end{cases}
\]

It follows that for \( R \) large either the first term (i) is at least as large as II or the second term (ii) is of the order of II. Thus (i) + (ii) is at least as large as II which is much bigger than (iii). This proves Theorem (2.13)(b). The others are easier.

We have also derived the fact that the essential contribution to \( \phi(x) = \phi(R, \xi_0) \) for \( R \) large \( (\geq R(\varepsilon)) \) and \( \mu \)-almost all \( \xi_0 \) comes from the part of the integral with \( d(\xi, \xi_0) \leq \varepsilon \) for any \( \varepsilon > 0 \). This is useful for Theorem (2.14).

We now write out

\[
\phi_1(x) = \int_{\xi} \chi(\xi) \phi(x, \xi, a) \, d\mu, \quad \phi_2(x) = \int_{\xi} 1 \cdot \phi(x, \xi, a) \, d\mu,
\]

where \( \mu = \mu(p, \phi_1) \) and \( \chi(\xi) = d\mu(p, \phi_1)/d\mu(p, \phi_2)(\xi) \). By the above for \( \mu \)-almost all \( \xi_0 \) and for \( R \) large we only need consider the integrals for \( d(\xi, \xi_0) < \varepsilon \).

Now consider a set \( A \) of \( \xi \) of positive \( \mu \)-measure where \( \chi(\xi) \) is approximately \( a \). For \( x = (R, \xi_0) \), \( \phi(x, \xi, a) \) only depends on \( d(\xi, \xi_0) \), as indicated above. Moreover, \( \phi(x, \xi, a) \) only varies up to a constant near 1 in ratio on annuli of a definite shape around \( \xi_0 \) (again, from the above).
For each $\xi_0$ in a subset $B \subset A$ of full $\mu$-measure we can choose $\varepsilon$ so that if we divide the $\varepsilon$-disk about $\xi_0$ into concentric annuli of (relative) constancy for $\phi(x, \xi, \alpha)$ ($x = (R, \xi_0)$, $R > R(\varepsilon)$) each of these annuli will be mostly filled (relative to $\mu$) by points of $A$, and the $\mu$-integral of $\chi$ on each is approximately $a$. This follows from Lebesgue density and differentiation. Then we see that $\phi_1(x)$ and $\phi_2(x)$ are sums of terms in approximate ratio $a$ which is approximately $\chi(\xi_0)$. These sets $A$ fill up $\mu$. This proves Theorem (2.14).

Now we turn to the proof of Theorem (2.15). Let $A$ be a set of positive $\mu$-measure so that $\phi(R, \xi) \leq e^{(\sigma + \varepsilon)R}$ for $\varepsilon > 0$ and $R > R(\xi, \varepsilon)$. Fixing $\varepsilon$ we can make $R(\xi, \varepsilon)$ independent of $\xi$ by reducing $A$ a little to $B$. Write $\delta = \sigma + \varepsilon$ and $r = e^{-R}$. Referring to the decomposition of the integral for $\phi(x)$ above, we deduce that the first term is $\leq e^{\delta R}$. Thus $\mu(B, r) \approx r^{\alpha - \delta}$ for any $\xi_0$ in $B$. For any covering of $B$ by balls of radius $r_i$ centered at $\xi_i$ in $B$ we have

$$0 < \mu(B) \approx \sum_i \mu(\xi_i, r_i) \approx \sum_i r_i^{\alpha - \delta}.$$  

Thus the Hausdorff $(\alpha - \delta)$-measure of $B$ is positive. So the Hausdorff dimension of $A \supset B$ is $\geq \alpha - \delta = \alpha - \sigma + \varepsilon$ for every $\varepsilon > 0$. This proves Theorem (2.15).

11. Proof of Theorems (2.17), (2.19), (2.21), (2.23), and (2.24)

If $M = \mathbb{H}^{d+1}/\Gamma$, then $p^M(x, y)$ is just $\sum_{\gamma \in \Gamma} p^M_\gamma(x, \gamma y)$, where $x, y$ lie in $\mathbb{H}^{d+1}$ over $x, y$. Thus $g^M(x, y) = \sum_{\gamma \in \Gamma} g_\gamma(x, \gamma y)$. So if $x$ is not on the $\Gamma$ orbit of $y$, then $g^M(x, y)$ has the order of the Poincaré series $\sum_{\gamma} \exp(-\alpha d(x, \gamma y))$ by Theorem (2.16), $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{2}d^2)^{1/2}$. Thus $g^M(x, y) < \infty$ for $x \neq y$ if $\alpha > \delta(\Gamma)$ and $g^M(x, y) = \infty$ for $a < \delta(\Gamma)$ when $\delta(\Gamma) \geq \frac{1}{2}d$. This means $\lambda_0(M) = \delta(\Gamma)(\delta(\Gamma) - d)$ if $\delta(\Gamma) \geq \frac{1}{2}d$ by Theorem (2.6). Otherwise $\lambda_0(M) = -\frac{1}{4}d^2$, since $\lambda_0(M) \geq -\frac{1}{4}d^2$ by Theorem (2.1) and Theorem (2.11). This proves Theorem (2.17).

Theorem (2.19) is partially proved in [26] generalizing [20], namely $\delta(\Gamma)$ is the minimum power (which is achieved) for a measure satisfying (2.1) [26, §2].

If $\delta > \delta(\Gamma)$, put a Dirac mass at each point of the orbit $\gamma(y)$ of a point $y$ in the open ball model $B^{d+1}$ of $\mathbb{H}^{d+1}$ with weight $|\gamma'y|^2$. A measure of finite mass results because the Poincaré series converges at $\delta > \delta(\Gamma)$. This measure satisfies (2.1) but is not supported on $\mathbb{S}^d$. The set of measures of bounded mass satisfying (10.1) supported in the closed ball is a closed set. Thus let $y$ approach infinity in a fundamental domain and take a limit to prove Theorem (2.19)(i).
To prove Theorem (2.19)(ii) we merely let \( y \) approach a limit point staying in one fundamental domain (then all the mass approaches the limit set) and this is possible unless \( \Gamma \) is geometrically finite without cusps.

In that case there is only a measure of exponent \( \delta(\Gamma) \), i.e. \( \delta = \delta(\Gamma) \), and this is Hausdorff measure by [26, §3]. This completes the proof of Theorem (2.19).

To prove Theorem (2.21)(i) we merely quote [28], which proves \( \delta(\Gamma) = \) the Hausdorff dimension of the limit set for geometrically finite groups, and apply Theorem (2.17). Part (ii) also follows from [28]. Thus Theorem (2.21) is proved. The corollary is a local consequence.

Theorem (2.23) follows from Theorem (2.17) and [26, Corollary 27]. Theorem (2.24) follows from Theorem (2.21) and the definitions (see [14]).

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**References**


