J. DIFFERENTIAL GEOMETRY
25 (1987) 327–351

RELATED ASPECTS OF POSITIVITY IN RIEMANNIAN GEOMETRY

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1. Introduction

There are several numerical functions which can be related in the geometric context of Riemannian manifold, especially those which are complete and have constant negative curvature. They are:

(i) The Hausdorff dimension $D(X)$ of a closed set X in Euclidean space.

(ii) The point λ_0 of the L^2-spectrum nearest to zero for the Friedrich extension of a semidefinite symmetric operator Δ.

(iii) The critical exponent $\delta(\Gamma)$ of the Poincaré-Dirichlet series of a discrete group Γ of Moebius transformations of S^d.

(iv) The parameter of the spherical complementary series of irreducible representations of $0(n,1)$, especially $PSl(2,\mathbf{R})$ and $PSl(2,\mathbf{C})$.

(v) The exponential rate of transience of a positivity preserving Markoff operator P, equivalently the point nearest to zero in the 'positive spectrum' of a Markoff operator P, i.e., λ-potential theory.

(vi) The entropy of an ergodic measure preserving flow.

A rich supply of examples is given by groups Γ of non-Euclidean motions of $\mathbf{H}^{d+1}$ having *finite sided* fundamental domains in $\mathbf{H}^{d+1}$ whose Poincaré limit set in S^d has Hausdorff dimension $D > \frac{1}{2}d$. Then:

(1) The Hausdorff measure μ of the limit set is finite and positive (when the ranks of any cusps are at most D).

(2) μ on the λ-Martin boundary S^d of $\mathbf{H}^{d+1}$ defines a positive λ-eigenfunction ϕ_μ for the Laplacian which is Γ-invariant, where $\lambda = D(d-D)$. (In the situation of (1), $\phi_\mu(p)$ has the following nice geometrical interpretation: It is the total Hausdorff mass of the limit set as viewed from p.)

(3) ϕ_μ belongs to $L^2(\mathbf{H}^{d+1}/\Gamma)$ and is the lowest eigenfunction or ground state.

Received August 9, 1985 and, in revised form, January 22, 1986.

(4) ϕ_μ determines an irreducible unitary representation for $0(d+1,1)$ with parameter D, which is also the critical exponent $\delta(\Gamma)$ of the Poincaré series of Γ.

(5) ϕ_μ is the unique positive λ-eigenfunction of Δ, Γ-invariant, and λ is the value nearest to zero. The heat kernel $p_t(x, y)$ on $\mathbf{H}^{d+1}/\Gamma$ decays as $e^{\lambda t}$.

(6) μ determines an invariant measure for a geodesic flow of total mass $\int \varphi_\mu^2$ whose measure theoretic entropy (and this is also the topological entropy when there are no cusps) is D (see [25], [26], [27], and [28]).

In the first part of the paper we point out the general connections for Riemannian manifolds between (ii) and (v). In the second part we specialize to constant negative curvature and connect up (i), (iii), and (iv) with (ii) and (v). For (vi) see [28]. Most of these results are individually known to separate groups. It is our hope that the entire synthesis may have a certain general interest. (Note, in this the operator Δ is $\leqslant 0$.)

2. Statement of results

2.1. Riemannian manifolds: Definition of $\lambda_0(M)$. Let M be an open connected Riemannian manifold without boundary. Define the real number λ_0 in $(-\infty, 0]$ as the negative of the infimum of $\int_M |\text{grad}\,\phi|^2/\int_M |\phi|^2$ over smooth functions ϕ on M with compact support. First, we take the λ-potential theory approach to $\lambda_0(M)$. Say that a smooth function ϕ on M is *λ-harmonic* if $\Delta\phi = \lambda\phi$, where Δ is the Laplacian.

Theorem (2.1). *For each $\lambda \geqslant \lambda_0$ there are positive λ-harmonic functions on M. For each $\lambda < \lambda_0$ there are no positive λ-harmonic functions on M.*

Compare [6], [10], and [19].

Second, we take the Hilbert space approach to $\lambda_0(M)$. There is a canonical self-adjoint operator (also denoted Δ) on $L^2(M)$ extending the Laplacian on smooth function with compact support. If M is complete, all self-adjoint extensions agree and Δ is unique [12]. In the general case we take for Δ the infinitesimal generator of the (minimal) heat semigroup, $f(x, t) = \int_M p_t(x, y) f(y)\, dy$. Here the symmetric positive kernel $p_t(x, y)$ is *defined* to be the supremum (an increasing limit) over all smooth compact subregions with boundary $(M_\alpha, \partial M_\alpha)$ of the fundamental solutions $p_t^\alpha(x, y)$ for the heat equation in M_α vanishing on the boundary ∂M_α.

$$p_t(x, y) = \sup_\alpha p_t^\alpha(x, y) \quad \text{and} \quad \left(\frac{\partial}{\partial t} - \Delta_x\right) p_t(x, y) = 0$$

(compare [7]).

Theorem (2.2). *The closed L^2-spectrum of Δ contains λ_0 and is contained in the negative ray* $(-\infty, \lambda_0]$.

Compare (Fukushima and Stroock [11]), where the minimal heat kernel Δ construction above is shown to be equal to the Friedrich abstract minimal self-adjoint extension of the Laplacian.

Corollary (2.3). *For $\lambda > \lambda_0$, the symmetric kernel $\int_0^\alpha e^{-\lambda t} p_t(x, y)\,dt$ defines a bounded operator on L^2, namely* $(\Delta - \lambda)^{-1}$.

Combining Theorems (2.1) and (2.2) we have the following spectral picture for any open Riemannian manifold: $\lambda_0 \leqslant 0$ and λ_0 separates the L^2-spectrum from the "positive spectrum": $\{\lambda \mid \exists\phi < 0, \Delta\phi = \lambda\phi\}$.

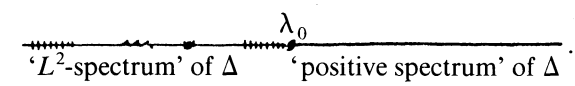

FIGURE 1

Example (2.4). For M the real line (or Euclidean space) and $\lambda_0 = 0$, the functions $e^{\alpha x}$, α real, are α^2-harmonic and $\{e^{-i\alpha x}\}$ are virtual L^2 eigenfunctions belonging to $-\alpha^2$ as continuous spectrum.

Example (2.5). For M the hyperbolic plane and $\lambda_0 = -\frac{1}{4}$, the positive λ-harmonic functions for $-\frac{1}{4} \leqslant \lambda \leqslant 0$ are related to the complementary series of $Sl(2, \mathbf{R})$ (see §2.3), and the virtual L^2 eigenfunctions, as continuous spectrum on $(-\infty, -\frac{1}{4}]$, are related to the principal series of $Sl(2, \mathbf{R})$.

Thirdly, we have the Markoff process approach to λ_0. We say that λ *belongs to Green's region* of M if for some pair (x, y), $x \neq y$.

$$\int_0^x e^{-\lambda t} p_t(x, y)\,dt < \infty.$$ [1]

A variant of a classical proposition (see §5) is that for λ in the Green's region the integral converges for all pairs (x, y), $x \neq y$, and defines the λ-Green's function $g_\lambda(x, y)$ which is locally integrable and satisfies

$$(\Delta_x - \lambda) g_\lambda(x, y) = \text{dirac mass at } y.$$

So for each y, $g_\lambda(x, y)$ defines a positive λ-harmonic function on $M \setminus \{y\}$.

Theorem (2.6). *For any open Riemannian manifold the Green's region consists of either* (i) *the open ray* (λ_0, ∞), *or* (ii) *the closed ray* $[\lambda_0, \infty)$.

[1] Note in this paper the Laplacian is a negative semi-definite operator because of the first figure.

In case (i), $\int_0^x e^{-\lambda_0 t} p_t(x, y)\, dt = \infty$, M is said to be λ_0-*recurrent*. In case (ii), $\int_0^\infty e^{-\lambda_0 t} p_t(x, y)\, dt < \infty$, M is said to be λ_0-*transient*.

Now we discuss situations in which positive λ_0-harmonic functions are unique (up to constant multiples).

Theorem (2.7) (*Recurrent case*). *If the Green's region is* (λ_0, ∞), *i.e.*, $\int_0^\infty e^{-\lambda_0 t} p_t(x, y)\!: dt = \infty$, *then the positive* λ_0-*harmonic functions are constant multiples of one another.*

Theorem (2.8) (*Square integrable case*). *Suppose the spectral measure of* Δ *has an atom at* λ_0. *Then the* λ_0 *eigenspace of* Δ *is one-dimensional and is generated by a* (*square integrable*) *positive* λ_0-*harmonic function* ϕ_0.

Also, the integral $\int_0^\infty e^{-\lambda_0 t} p_t(x, y)\, dy$ *diverges so* M *is* λ_0-*recurrent and any positive* λ_0-*harmonic function is square integrable and thus a multiple of* ϕ_0.

We note here the related statement: If any atom of the spectral measure of Δ is represented by a (square integrable) positive λ-harmonic function, then $\lambda = \lambda_0$ and this atom is situated at λ_0. This follows directly from Theorems (2.1) and (2.2).

Corollary (2.9). *If a complete manifold* M *possess a positive square integrable eigenfunction* ϕ *for* Δ, *then the eigenvalue is* $\lambda_0(M)$ *and* ϕ *is unique up to a constant multiple.*

2.2. Renormalization of random motion. Given any positive λ-harmonic function ϕ we can add to the usual random motion on M a force field or drift term $\operatorname{grad}\log\phi$. Then we have a biased random motion (the ϕ-process) corresponding to the second order operator $\Delta + 2\operatorname{grad}\log\phi$, which acts on functions by (cf. §8)

$$f \to \Delta f + 2\operatorname{grad}\log\phi \cdot \operatorname{grad} f.$$

The transition probabilities for the ϕ-process are

$$\left(e^{-\lambda t}\phi(y)/\phi(x)\right) p_t(x, y)\, dy.$$

When the ϕ-process preserves the constant function 1 we say that ϕ is *complete*. This amounts to the reproducing formula

$$\phi(x) = \int_M e^{-\lambda t} p_t(x, y)\phi(y)\, dy.$$

(The inequality $\geqslant$ is always true.) When ϕ is complete the ϕ-process also preserves the measure $\phi^2(y)\, dy$ (cf. §8 and [24]).

When there is only one positive λ-harmonic function up to a multiple we refer to the ϕ-process as the λ-process.

Theorem (2.10). *Suppose* M *is* λ_0-*recurrent* $(\int_0^\infty e^{-\lambda_0 t} p_t(x, y)\, dt = \infty)$. *Then the* λ_0-*process associated to the second order operator* $\Delta + \operatorname{grad}\log\phi_0$ *preserves the function* 1, *the measure* $\phi_0^2(y)\, dy$, *and is recurrent—almost every* *path of the λ_0-process starting from any point in M enters every set of positive measure infinitely often.*

In the square integrable case (Theorem (2.8)) the λ_0-process preserves a finite measure, $\phi_0^2(y)\,dy$.

2.3. Hyperbolic manifolds. Let M be the unique connected complete simply connected $(d+1)$-manifold of constant negative curvature $\mathbf{H}^{d+1}$. We recall the two kinds of examples of positive λ-harmonic functions on $\mathbf{H}^{d+1}$.

First consider a Borel set A in $\mathbf{H}^{d+1}$'s visual sphere at infinity S^d which has finite positive Hausdorff measure in dimension α. Define a positive $\alpha(\alpha - d)$-harmonic function ϕ_A on $\mathbf{H}^{d+1}$ by the rule: $\phi_A(x) =$ Hausdorff α-measure of A in the visual metric on S^d as viewed from x. (That ϕ_A is λ-harmonic follows from the discussion below.)

Second, given ξ in S^d choose stereographic projection of the ball model for $\mathbf{H}^{d+1}$ to the upper half-space model for $\mathbf{H}^{d+1}$ with $\xi \leftrightarrow \infty$. If y is the vertical coordinate, then $\phi(x, \alpha\xi) = (y(x))^\alpha$ is a positive $\alpha(\alpha - d)$-harmonic function on $\mathbf{H}^{d+1}$. (In these coordinates, $\Delta = y^2$ (Euclidean Δ) $+ (1-d)y\partial/\partial y$.)

Note that in these examples both α and $d - \alpha$ lead to the same eigenvalue $\lambda = \alpha(\alpha - d) = (d-\alpha)((d-\alpha) - d)$. Also λ is a minimum $-\frac{1}{4}d^2$ for $\alpha = \frac{1}{2}d$.

Theorem (2.11). (i) *For $\mathbf{H}^{d+1}$, $\lambda_0 = -\frac{1}{4}d^2$*, [17], [21].

(ii) *Fix $p \in \mathbf{H}^{d+1}$. Then every positive λ-harmonic function ϕ is uniquely expressible in terms of the $\phi(\cdot, \alpha, \xi)$,*

$$\phi(x) = \int_{S^d} \phi(x, \alpha, \xi)\, d\mu(p, \phi)(\xi),$$

where $\alpha = \frac{1}{2}d + (\lambda - \lambda_0)^{1/2}$, the $\phi(\cdot, \alpha, \xi)$ are normalized to be 1 at p, and $\mu(p, \phi)$ is a unique positive measure on S^d with total mass $\phi(p)$ [15].

The next two theorems concern the boundary measure $\mu(p, \phi)$ and its measure class for any positive λ-harmonic function ϕ. Let $\mu(p, \phi, R)$ be the measure on the sphere $S(p, R)$ of hyperbolic radius R centered at p, i.e.,

$$\mu(p, \phi, R) = 1/c_R \cdot (\phi \text{ restricted to } S(p, R)) \cdot \text{spherical measure},$$

where

$$c_R = \begin{cases} e^{-(d-\alpha)R} & \text{if } \lambda > -\frac{1}{4}d^2, \\ Re^{-(d/2)R} & \text{if } \lambda = -\frac{1}{4}d^2, \end{cases}$$

and $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$.

Theorem (2.12). *In the compactified space* $\mathbf{H}^{d+1} \cup S^d$, *the boundary measure* $\mu(p,\phi)$ *of Theorem* (2.11) *is constructed from* ϕ *as a weak limit of the* $\mu(p,\phi,R)$.

$$\lim_{R\to\infty} \mu(p,\phi,R) = \mu(p,\phi).$$

Now we consider radial limits, along hyperbolic rays (R,ξ) emanating from p, of a positive λ-harmonic function ϕ with $\phi(p)=1$.

Theorem (2.13). (a) *For* ξ *outside the closed support of* $\mu(p,\phi)$,

$$\phi(\xi,R) \sim e^{-\alpha R} \quad \text{as } R\to\infty.$$

(b) *For* $\mu(p,\phi)$*-almost all* ξ,

$$\phi(\xi,R) \geqslant \begin{cases} e^{-(d-\alpha)R} & \text{for } \lambda > -\frac{1}{4}d^2, \\ Re^{(-d/2)R} & \text{for } \lambda = -\frac{1}{4}d^2. \end{cases}$$

(c) *For all* ξ,

$$\phi(\xi,R) \leqslant e^{\alpha R} \quad \text{as } R\to\infty.$$

Again $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$.

Now we give a generalization of Fatou's theorem. Suppose ϕ_1 and ϕ_2 are positive λ-harmonic functions and $\mu(p,\phi_1)$ is absolutely continuous with respect to $\mu(p,\phi_2)$ with Radon-Nikodým derivative $\psi(\xi)$.

Theorem (2.14). *For* $\mu(p,\phi_2)$*-almost all* ξ

$$\lim_{R\to\infty} \phi_1(R,\xi)/\phi_2(R,\xi) = \psi(\xi).$$

In particular if $\phi_1 \leqslant \phi_2$, *then* $\mu(p,\phi_1) \leqslant \mu(p,\phi_2)$ *by Theorem* (2.12), *and the conclusion holds.*

Define the *exponential growth* of ϕ along a hyperbolic ray (R,ξ) from p in the direction ξ by

$$\limsup_{R\to\infty} \frac{\log\phi(R,\xi)}{R}.$$

By Theorem (2.13) this growth is always $\leqslant \alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$. Suppose the growth is smaller, $\leqslant \sigma$, for a set of directions $A \subset S^d$ of positive $\mu(p,\phi)$ measure.

Theorem (2.15). (i) *The Hausdorff dimension of A is at least*

$$\alpha - \sigma = \left(\tfrac{1}{2}d + \left(\lambda + \tfrac{1}{4}d^2\right)^{1/2}\right) - \sigma.$$

(ii) *In particular if* ϕ *is bounded, the Hausdorff dimension of any* $\mu(p,\phi)$*-positive set is at least* $\frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$.

We describe the behavior of the λ-Green's function $g_\lambda(x, y) = \int_0^x e^{-\lambda t} p_t(x, y)\, dt$ on $\mathbf{H}^{d+1}$, which is finite for $\lambda \in [\lambda_0, \infty)$ and only depends on $r = d(x, y)$ for r near ∞. It is convenient to include a description of the λ-*spherical function* $S_\lambda(x, y)$ which is by definition the unique (up to a multiple) positive λ-harmonic function of x in $\mathbf{H}^{d+1}$, spherically symmetric about y in $\mathbf{H}^{d+1}$. These two functions are solutions of the second order differential equation in the radius R which has regular singular points at $R = 0$ and $R = \infty$.

Theorem (2.16). *For $\lambda \geqslant \lambda_0$, $g_\lambda(x, y)$ and $S_\lambda(x, y)$ generate the two-dimensional space of spherically symmetric solutions of $(\Delta - \lambda) f = 0$ on $\mathbf{H}^{d+1} \setminus \{y\}$. The λ-Green's function $(\int_0^\infty e^{-\lambda t} p_t(x, y)\, dt)$ is the small (or recessive) solution near $R = \infty$, and the λ-spherical function $(\int_{S^d} \phi(x; \xi, \alpha)\, d\theta(\xi))$ is the small (or recessive) solution near $R = 0$.*

Thus if $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$, then $g_\lambda \sim$ const $\cdot\, e^{-\alpha R}$ near $R = \infty$, while $S_\lambda \sim$ const $\cdot\, e^{-(d-\alpha)R}$ near $R = \infty$, except when $\alpha = \frac{1}{2}d$ where $S_\lambda \sim$ const $\cdot$ $Re^{-(d/2)R}$ near $R = \infty$.

Now let Γ be any discrete group of hyperbolic isometries. If Γ has no torsion, then $\mathbf{H}^{d+1}/\Gamma$ is a complete Riemannian manifold with constant negative curvature to which the generalities of §2.1 apply. We have the generalized Elstrodt-Patterson theorem.

Theorem (2.17). *For $M = \mathbf{H}^{d+1}/\Gamma$, $\lambda_0(M)$ satisfies*

$$\lambda_0(M) = \begin{cases} -\frac{1}{4}d^2, & \text{if } \delta(\Gamma) \leqslant \frac{1}{2}d, \\ \delta(\Gamma)(\delta(\Gamma) - d), & \text{if } \delta(\Gamma) \geqslant \frac{1}{2}d, \end{cases}$$

where $\delta(\Gamma)$ is the critical exponent of Γ. (*Compare* [8], [29].)

Recall the *critical exponent* $\delta(\Gamma)$ is defined so that the Poincaré series of Γ,

$$g(x, y, s) = \sum_{\gamma \in \Gamma} \exp -(sd(x, \gamma y)),$$

converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$, where (x, y) is any pair of points in $\mathbf{H}^{d+1}$ and $d(x, y)$ is the hyperbolic distance.

Corollary (2.18) (*Of proof*). *If $M = \mathbf{H}^{d+1}/\Gamma$, M is λ_0-recurrent iff $\delta(\Gamma) \geqslant \frac{1}{2}d$ and the Poincaré series diverges at $s = \delta(\Gamma)$.*

Now λ-harmonic functions on M are just Γ-invariant λ-harmonic functions on $\mathbf{H}^{d+1}$. From the definition it follows that for any positive λ-harmonic function ϕ on $\mathbf{H}^{d+1}$ and for any isometry γ of $\mathbf{H}^{d+1}$.

$$\gamma^* \mu(p, \phi) = |\gamma'|^\alpha \mu(p, \phi \cdot \gamma),$$

where $|\gamma'|$ is the linear distortion of the visual metric on S^d as viewed from p, $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$ as before, and $\gamma^*\mu(\text{set}) = \mu(\gamma(\text{set}))$.

Thus if ϕ is invariant by Γ, then $\mu(p, \phi)$ on S^d satisfies

$$\gamma^*\mu = |\gamma'|^\delta \mu, \tag{2.1}$$

where $\delta = \alpha$ and $\gamma \in \Gamma$.

Thus Theorem (2.17) yields the existence of measures on S^d satisfying (2.1). Curiously, a bit more can be said about this question than the λ-potential theory implies. The following theorem generalizes earlier results of Patterson [29] and the author [26].

Theorem (2.19). (i) *If Γ is any discrete group of isometries of $\mathbf{H}^{d+1}$ (except for elementary parabolic or cocompact groups) there is a finite positive measure on S^d satisfying $\gamma^*\mu = |\gamma'|^\delta\mu$, $\gamma \in \Gamma$, iff $\delta \in [\delta(\Gamma), \infty)$.*

(ii) *We may further suppose that μ is concentrated on the limit set of Γ unless Γ is geometrically finite without cusps. In these latter cases (including cocompact groups) the only such measure on the limit set is the Hausdorff measure in dimension $\delta(\Gamma)$.*

The *limit set of* Γ is by definition the set of cluster points in S^d of any Γ orbit in $\mathbf{H}^{d+1}$. The condition *geometrically finite without cusps* means that Γ has a finite sided fundamental domain in $\mathbf{H}^{d+1}$ which does not touch the limit set.

Remark (2.20). For the elementary parabolic groups there are point measures in S^d satisfying (2.1) for any δ in $[0, \infty)$ even though $\delta(\Gamma) = \frac{1}{2}$(rank of parabolic subgroup) > 0.

We mention two more theorems relating the λ-potential theory of $M = \mathbf{H}^{d+1}/\Gamma$ and the Hausdorff geometry at infinity.

Theorem (2.21). (i) *If Γ is geometrically finite and $M = \mathbf{H}^{d+1}/\Gamma$, then*

$$\lambda_0(M) = \begin{cases} -\frac{1}{4}d^2, & D \leqslant \frac{1}{2}d, \\ D(D-d), & D \geqslant \frac{1}{2}d, \end{cases}$$

where D is the Hausdorff dimension of the limit set.

(ii) *M has a square integrable positive λ_0-harmonic function iff $D > \frac{1}{2}d$. M is λ_0-recurrent iff $D \geqslant \frac{1}{2}d$.*

Corollary (2.22). *Let $M = \mathbf{H}^{d+1}/\Gamma$, where Γ is geometrically finite. Then whether or not the Hausdorff dimension of the limit set belongs to $(0, \frac{1}{2}d)$ and if not its exact value in $[\frac{1}{2}d, d]$ is determined by the λ-potential theory of M.*

Any discrete group of isometries of the hyperbolic plane $\mathbf{H}^2$ is a union of geometrically finite groups. This allows a general result from [26].

Theorem (2.23). *For any complete connected hyperbolic surface S let D denote the Hausdorff dimension of the set of those geodesics emanating from any fixed point in S which returns infinitely often to any bounded neighborhood of that* *point. Then* $\lambda_0(S)$ *satisfies*

$$\lambda_0(S) = \begin{cases} -\frac{1}{4}, & D \leqslant \frac{1}{2}, \\ D(D-1), & D \geqslant \frac{1}{2}. \end{cases}$$

Recall $G(d)$ denotes the group of proper motions of $\mathbf{H}^{d+1}$. Then $G(1) = Pl(2, \mathbf{R})$ and $G(2) = PSl(2, \mathbf{C})$.

Now Theorem (2.21) allows a canonical geometric interpretation of the complementary series in terms of hyperbolic manifolds $\mathbf{H}^{d+1}/\Gamma$ and the Hausdorff geometry of the limit sets of the discrete groups Γ.

Theorem (2.24). *Let* ϕ_0 *denote the square integrable positive* λ_0*-harmonic function on* $M = \mathbf{H}^{d+1}/\Gamma$, *where* Γ *is geometrically finite and the Hausdorff dimension of the limit set* $D = \delta(\Gamma) > \frac{1}{2}d$. *Then the linear span of the* $G(d)$*-orbit of* ϕ_0 *in* $L^2(G(d)/\Gamma)$ *generates the member of the complementary series labeled by* $\lambda_0(M) \in (-\frac{1}{4}d^2, 0)$.

For example, if Γ *has no cusps (or all cusps have rank* $\leqslant D$) *then* $\phi_0(p)$, *the K-invariant vector, is just the function on* $\mathbf{H}^{d+1}$ *which assigns the Hausdorff D-measure of the limit set of* Γ *calculated in the metric as viewed from p.*

Remark (2.25). There are examples where deformations of one Γ make λ_0 cover the entire (spherically symmetric) complementary series [25], [3].

3. Preliminaries: Compact manifolds with smooth boundary

Let M_α be a compact manifold with smooth boundary. Let $p_t^\alpha(x, y)$ be the fundamental solution of the heat equation in M_α vanishing on ∂M_α (cf. [22]). The infinitesimal generator of the semigroup

$$f(x, t) = \int_{M_\alpha} p_t^\alpha(x, y) f(y)\, dy$$

defines a self-adjoint operator Δ on $L^2(M_\alpha)$ extending the Laplacian acting on smooth functions vanishing near the boundary [22].

By the compactness of M_α there is a discrete set of eigenvalues for Δ,

$$\cdots < \lambda_2^\alpha < \lambda_1^\alpha < \lambda_0^\alpha < 0,$$

and a complete basis of L^2 consisting of eigenfunctions vanishing on the boundary.

Since $|\lambda_0^\alpha|$ is the infimum of $\int_{M_\alpha} |\operatorname{grad} \phi|^2 / \int_{M_\alpha} |\phi|^2$ over smooth functions vanishing near the boundary, any eigenfunction ϕ_0 belonging to λ_0^α does not change sign (see §8 for an alternative argument). It follows that λ_0^α has multiplicity 1 and ϕ_0 is unique up to a constant multiple.

Since one may write an absolutely convergent eigenexpansion for $p_t^\alpha(x, y)$,

$$p_t^\alpha(x, y) = \sum e^{\lambda_n^\alpha t} \phi_n^\alpha(x) \phi_n^\alpha(y) \tag{3.1}$$

[22], one has

$$\lim_{t \to \infty} e^{-\lambda_0^\alpha t} p_t^\alpha(x, y) = \phi_0^\alpha(x) \phi_0^\alpha(y), \tag{3.2}$$

where ϕ_0^α is the unique positive normalized zeroth eigenfunction.

From the probabilistic interpretation [18] of $p_t^\alpha(x, y)\,dy$ as the probability density of endpoints of random paths starting at x which have not hit the boundary before time t, one has from (3.2) that the probability of starting from x and hitting the boundary ∂M_α by time t is asymptotically 1 like

$$1 - \text{const}\, e^{(\lambda_0^\alpha) t}. \tag{3.3}$$

Now recall the Dirichlet problem for M_α. If f is a continuous function on ∂M_α, then the harmonic extension of f inside M_α may be written

$$f(x) = \int_{\partial M_\alpha} f(\xi)\, d\mu_\alpha(x, \xi), \tag{3.4}$$

where $\mu_\alpha(x, \xi)$ is the probability measure associated to hitting the boundary with random paths starting from x.

Now weight the hitting probability by $e^{-\lambda\tau}$, where τ is the hitting time and λ is any number $> \lambda_0^\alpha$. By (3.3) the resulting measure $\mu_\alpha^\lambda(x, \xi)$ is well defined and finite. Again if f is a continuous function on the boundary, then

$$f(x) = \int_{\partial M_\alpha} f(\xi)\, d\mu_\alpha^\lambda(x, \xi) \tag{3.5}$$

defines a smooth λ-harmonic function in M_α with boundary values f. The classical proof of (3.4) may be modified to give (3.5) replacing Δ by $\Delta - \lambda$.

Now recall that the generalized Poisson measures $\mu_\alpha(x, \xi)$ of (3.4) are equivalent for various x and that for fixed x_0 in M_α the ratio $d\mu_\alpha(x, \xi)/d\mu_\alpha(x_0, \xi) = \psi_\alpha(x, \xi)$ is a positive harmonic function for ξ fixed (which is zero on $\partial M_\alpha \setminus \{\xi\}$ and has a pole at ξ). Similarly $d\mu_\alpha^\lambda(x, \xi)/d\mu_\alpha^\lambda(x_0, \xi) = \psi_\alpha^\lambda(x, \xi)$ is a positive λ-harmonic function on M_α for ξ fixed (which is zero on $\partial M_\alpha \setminus \{\xi\}$ and has a pole at ξ). (See §2.3 for examples.)

This shows the Harnack principle for positive harmonic functions is also valid for positive λ-harmonic functions, $\lambda > \lambda_0^\alpha$. Namely, write (3.5) as

$$(3.6) \qquad f(x) = \int_{\partial M_\alpha} f(\xi) \frac{d\mu_\alpha^\lambda(x, \xi)}{d\mu_\alpha^\lambda(x_0, \xi)} \, d\mu_\alpha^\lambda(x_0, \xi)$$

$$= \int_{\partial M_\alpha} f(\xi) \psi_\alpha^\lambda(x, \xi) \, d\mu_\alpha^\lambda(x_0, \xi),$$

showing that the values of f around x are fixed convex combinations $(f(\xi)\, d\mu_\alpha^\lambda(x_0, \xi))$ of values $(\psi_\alpha^\lambda(x, \xi))$ which only vary in a bounded ratio.

4. Proof of Theorem (2.1)

Now consider the directed set of all compact connected regions $M_\alpha \subset M$ with smooth boundary. Since λ_0^α (of §3) is the negative of the infimum over smooth functions supported on interior M of $\int_{M_\alpha} |\text{grad}\, \phi|^2 / \int_{M_\alpha} |\phi|^2$, the number λ_0 defined in the introduction clearly satisfies

$$\lambda_0 = \sup_\alpha \lambda_0^\alpha,$$

and $\lambda_0 > \lambda_0^\alpha$ for all α.

Then by §3 there are positive λ-harmonic functions on M_α for any $\lambda \geqslant \lambda_0 > \lambda_0^\alpha$. By the Harnack principle described in §3 we have compactness with respect to uniform convergence on compact sets for those positive λ-harmonic functions which are $\leqslant 1$ at a fixed point x_0. We can form convergent subsequences of those defined for an exhaustion of M by M_α and thereby prove the first part of Theorem (2.1).

The second part of Theorem (2.1) follows from the fact that a positive λ-harmonic function f continuous on M_α satisfies

$$(4.1) \quad f(x) = \int_{M_\alpha} e^{-\lambda t} p_t^\alpha(x, y) f(y) \, dy + \int_p e^{-\lambda \tau} \, d(\text{Wiener measure}),$$

where p is the set of paths which hit ∂M_α at $\tau < t$. So

$$f(x) \geqslant \int_{M_\alpha} e^{-\lambda t} p_t^\alpha(x, y) f(y) \, dy.$$

This shows $\lambda \geqslant \lambda_0^\alpha$ using (3.2) and completes the proof of Theorem (2.1).

5. The Green's region and λ-superharmonic functions

Consider the function $g_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y)\, dt$ and suppose $g_\lambda(x, y)$ is finite for one pair $x \neq y$. From the definition, $g_\lambda(x, y)$ is symmetric and as a function of x it is

(1) the increasing limit of continuous functions (and so lower semicontinuous, $f(x) \leqslant \lim_{x_i \to x} f(x_i)$),

(2) decreased pointwise by at least the factor $e^{\lambda t}$ by the heat semigroup, $f(x,t) = \int_M p_t(x, y) f(y)\, dy$. Namely, $f(x, t) \leqslant e^{-\lambda t} f(x)$.

Functions of x satisfying (1) and (2) (and not identically $+\infty$) are called λ-*superharmonic*. So if λ belongs to the Green's region there is a λ-superharmonic function ($g_\lambda(x, y)$ for each y).

Conversely, suppose f is λ-superharmonic and let P_t^λ denote $e^{-\lambda t}$ (heat operator). We apply the operator equation

$$\int_0^T P_s^\lambda\, ds \frac{\mathrm{Id} - P_t^\lambda}{t} = \frac{1}{t}\int_0^t P_s^\lambda\, ds - \frac{1}{t}\int_T^{T+t} P_s^\lambda\, ds \tag{5.1}$$

to f and deduce using (1) and (2) that either

$$P_t^\lambda f = f \quad \text{for all } x, \tag{5.2}$$

or λ belongs to the Green's region.

Using the fact that for smooth functions of compact support ϕ

$$\frac{\mathrm{Id} - P_t^\lambda}{t}\phi \to -(\Delta - \lambda)\phi, \tag{5.3}$$

uniformly on compact sets as $t \to 0$, one obtains by duality that a λ-superharmonic function (which is locally integrable by $f \geqslant P_t^\lambda f$) satisfies

$$\lim_{t \to 0} \frac{f - P_t^\lambda f}{t} = -(\Delta - \lambda) f, \tag{5.4}$$

in the sense of distributions. Thus $-(\Delta - \lambda)f$ is a positive Radon measure approximated by $((f - P_t^\lambda f)/t)\, dy$, whenever f is λ-superharmonic.

Calculating the latter for $g_\lambda(x, y)$ (as a function of x for y fixed) yields

$$\frac{\mathrm{Id} - P_t^\lambda}{t} g_\lambda(x, y) = \frac{1}{t}\int_0^t e^{-\lambda t} p_s(x, y)\, ds$$

which approaches the Dirac mass at y as $t \to 0$. A corollary is that $g_\lambda(x, y)$ is finite for all $x \neq y$ and defines a positive λ-harmonic function on $M \setminus \{y\}$.

Another corollary is that if λ belongs to the Green's region, then for every compact K in M

$$\lim_{T \to \infty} e^{-\lambda t} \int_K p_T(x, y)\, dy = 0. \tag{5.5}$$

To see this choose $\varepsilon_i \to 0$ and $T_i \to \infty$, write

$$g(x, y) = \lim_{T_i \to \infty, \varepsilon_i \to 0} \int_{\varepsilon_i}^{T_i} e^{-\lambda t} p_t(x, y)\, dt,$$

and use the heat equation to calculate $(\Delta_x - \lambda)(g_\lambda(x, y))$. One gets two terms, the one near zero converges to the right answer, the Dirac mass at y, so the other one corresponding to ∞ must go to zero. Since the convergence is that of radon measures, (5.5) results.

Besides the Green's function, positive λ-harmonic functions also provide examples of λ-superharmonic functions. This follows using (4.1) repeatedly,

$$p_t(x, y) = \sup_\alpha p_t^\alpha(x, y) \quad \text{and} \quad M = \bigcup_\alpha M_\alpha.$$

More precisely, (4.1) shows that (λ_0, ∞) is contained in the Green's region because the second part (5.2) must hold for a positive λ_0-harmonic function whenever the λ of (4.1) belongs to (λ_0, ∞).

Now if $\lambda < \lambda_0$, then $\lambda < \lambda_0^\alpha$ for some α and if λ belongs to the Green's region, (5.5) implies $\int_K e^{-\lambda_0^\alpha t} p_t(x, y)\, dy \to 0$ as $t \to \infty$ contradicting (3.3). Thus the Green's region does not contain λ and must consist of either $[\lambda, \infty)$ or (λ_0, ∞). This proves Theorem (2.6).

6. The L^2-spectrum of Δ and the proof of Theorem (2.2)

Using the spectral theorem and the positivity of $p_t(x, y)$ one sees immediately that if the interval $[\lambda, \infty)$ does not intersect the L^2-spectrum of Δ (the infinitesimal generator of the semigroup $f(x, t) = \int_M p_t(x, y) f(y)\, dy$), then the bounded operator on L^2, $(\Delta - \lambda)^{-1}$, is represented by the positive kernel $\int_0^\infty e^{-\lambda t} p_t(x, y)\, dy$. Applying the operator to a positive function with compact support shows that $\int_0^\infty e^{-\lambda t} p_t(x, y)\, dy$ is finite a.e. Thus $[\lambda, \infty)$ is contained in the Green's region. So the entire component of the complement of the spectrum containing the positive reals is contained in the Green's region.

For the other inequality required by Theorem (2.2) consider the L^2-norm of $P_t f = \int_M p_t(x, y) f(y)\, dy$. This is the square root of

$$\int_M \left(\int p_t(x, y_1) f(y_1)\, dy_1 \int p_t(x, y_2) f(y_2)\, dy_2 \right) dx.$$

Thus,

$$\| P_t f \|_{L^2} = \left(\int_{M \times M} p_{2t}(y_1, y_2) f(y_1) f(y_2)\, dy_1\, dy_2 \right)^{1/2}, \tag{6.1}$$

by the semigroup equation for $p_t(x, y)$.

Now consider a positive, bounded, measurable f, with support contained in a compact K in the interior of M. By (5.5) for each y_2, $e^{-\lambda t}\int p_t(x, y_1)f(y_1)\,dy_1 \to 0$ as $t \to \infty$ if λ belongs to the Green's region. For a set A of y_2's in K of almost full measure this convergence is uniform. Thus if g is f times the characteristic function of A, we have, by (6.1), that the L^2-norm of $P_t g$ times $e^{-\lambda t}$ goes to zero as $t \to \infty$. The linear span of these g is dense in L^2. It follows the L^2-spectrum of Δ cannot have points greater than λ, for then there would be elements h in L^2 so that the L^2-norm of $P_t h$ would not decrease as fast as $e^{\lambda t}$. This proves Theorem (2.2).

The corollary to Theorem (2.2) is explained by the first paragraph of this section.

7. On the uniqueness of positive λ_0-harmonic functions (Proofs of Theorems (2.7) and (2.8))

Suppose the convex cone of positive λ_0-harmonic functions is not a single ray. The base of this cone $\{\phi | \phi(x_0) = 1\}$ is convex, metrizable, and compact in the topology of uniform convergence on compact sets by the Harnack principle of §3. Let f and g be two different extreme points of this compact convex set so that $f \leqslant g$ and $g \leqslant f$ are both false and form $\phi = \min\{f, g\}$.

Let $P_t^{\lambda_0}$ be the operator of §5. From (4.8) it follows that $P_t^{\lambda_0} f \leqslant f$ and $P_t^{\lambda_0} g \leqslant g$. Thus by positivity of $P_t^{\lambda_0}$, $P_t^{\lambda_0}\phi \leqslant \phi$ so ϕ is λ_0-superharmonic (§5). Since ϕ is not smooth ϕ cannot be λ_0-harmonic. (There is a transversality detail here which can be treated using multiples of f and g if necessary.) Thus, $P_t^{\lambda_0}\phi \neq \phi$ for some t and the second case of (5.2) must hold. Thus λ_0 belongs to the Green's region, i.e. $\int_0^\infty e^{-\lambda_0 t} p_t(x, y)\,dt < \infty$. This proves Theorem (2.7).

Now suppose there is an atom at λ_0 for the spectral measure of Δ on L^2. Since $\Delta - \lambda_0$ is the infinitesimal generator of $P_t^{-\lambda_0}$ we must have $P_t^{\lambda_0}\phi = \phi$ for ϕ in the λ_0 eigenspace of Δ. In particular, $\|P_t^{\lambda_0} g\|$ does not approach zero as $t \to \infty$ for a dense set of L^2. Thus by (5.5) λ_0 is not in the Green's region. This proves the second part of Theorem (2.8).

Now we give a proof that any ϕ in L^2 satisfying $P_t^{\lambda_0}\phi = \phi$ cannot change sign. By Theorem (2.2), $P_t^{\lambda_0}$ is a contraction on L^2, so $\|P_t^{\lambda_0}|\phi|\|_2 \leqslant \||\phi|\|_2$ where $|\phi|$ is the absolute value of ϕ. On the other hand,

$$|\phi(x)| = |P_t^{\lambda_0}\phi(x)| \leqslant P_t^{\lambda_0}|\phi|(x),$$

so $(|\phi|(x))^2 \leqslant (P_t^{\lambda_0}|\phi|(x))^2$. Combining these two gives $|\phi|(x) = P_t^{\lambda_0}|\phi|(x)$ a.e.

If ϕ is not entirely negative, at a generic point where $\phi(x) > 0$ we have

$$\phi(x) = \int_M e^{-\lambda_0 t} p_t(x, y) \phi(y) \, dy,$$

$$\phi(x) = |\bar{\phi}|(x) = \int_M e^{-\lambda_0 t} p_t(x, y) |\phi|(y) \, dy.$$

So $\phi = |\phi|$ a.e. and ϕ must be entirely positive.

Since any ϕ does not change sign no two can be orthogonal in L^2. This completes the proof of Theorem (2.8).

8. The ϕ-process and completeness of λ-harmonic functions (Proof of Theorem (2.10))

It is formal that the operator defined on functions by the kernel $e^{-\lambda t}\phi(y)/\phi(x) p_t(x, y)$ and on measures by duality preserves the function 1 and the measure $\phi^2(y)\,dy$ iff $\phi(x) = \int_M e^{-\lambda t} p_t(x, y)\phi(y)\,dy$ (i.e. ϕ is complete in the terminology of §2.2).

The differential operator or infinitesimal generator associated to this diffusion operator is $[\phi]^{-1}(\Delta - \lambda)[\phi]$, where $[\phi]$ denotes the multiplication operator by ϕ. Thus

$$\begin{aligned}
[\phi]^{-1}(\Delta - \lambda)[\phi] f &= \phi^{-1}(\Delta - \lambda)\phi f \\
&= \phi^{-1}((\Delta\phi)\cdot f + \phi \cdot \Delta f + 2 \operatorname{grad}\phi \cdot \operatorname{grad} f - \lambda\phi f) \\
&= \Delta f + 2 \operatorname{grad}\log\phi \cdot \operatorname{grad} f,
\end{aligned}$$

since $\Delta\phi = \lambda\phi$.

If M is λ_0-recurrent and ϕ_0 is the unique positive λ_0-harmonic function (up to a multiple), then by (5.2) we must have $\phi_0(x) = \int_M e^{-\lambda_0 t} p_t(x, y)\phi_0(y)\,dy$, namely the first of (5.2) holds. For otherwise, by the second of (5.2), λ_0 belongs to the Green's region. This proves all but the last part of Theorem (2.10).

To prove recurrence we simply check the criterion for recurrence that the Green's function of the process is identically $+\infty$. For the ϕ-process the Green's function is $\int_0^\infty e^{-\lambda_0 t}\phi(y)/\phi(x) p_t(x, y)\,dt$ which equals $+\infty$ since the $\phi(y)/\phi(x)$ factor does not matter. This proves Theorem (2.10).

Now let us discuss the question of completeness for λ-harmonic functions. We will give several arguments for the existence of complete λ-harmonic functions which depend on auxiliary hypotheses.

Argument (8.1) (Fixed point property). Let $\mathscr{C}_\lambda$ note the convex cone of positive λ-superharmonic functions. The heat semigroup operates on $\mathscr{C}_\lambda$. Using *compactness* of the base of $\mathscr{C}_\lambda$ *and continuity of* P_t (if true simultaneously) we have, by the fixed point theorem, fixed rays in $\mathscr{C}_\lambda$. Taking the minimum λ, namely λ_0, the equation $P^t\phi = c\phi$ implies $c = e^{-\lambda_0 t}$ and we arrive at a *complete positive* λ_0-harmonic function. (I am indebted to Dan Stroock for pointing out that a topology making $\mathscr{C}_\lambda$ have a compact base and P_t continuous for a general Riemannian manifold is not obvious.)

Argument (8.2) (Minimal λ-harmonic functions). Let $\mathscr{H}_\lambda$ denote the convex cone of nonnegative λ-harmonic functions. The base of $\mathscr{H}_\lambda$ is compact by the Harnack principle of §3. Suppose the heat semigroup preserves $\mathscr{H}_\lambda$ or that even $\mathscr{H} = \mathscr{H}_\lambda \cap P_t\mathscr{H}_\lambda \neq 0$ is a nontrivial convex cone with a compact base. Let f lie in an extreme ray of $\mathscr{H}$ and let $f^0 = P_t^\lambda f$. Then $f \geqslant f^0$ by (4.1) and f^0 belongs to $\mathscr{H}$. Now $g = f - f^0$ is nonnegative and λ-harmonic. If $f = P_t^\lambda h$, then $g = P_t^\lambda(h - f)$ so g belongs to $\mathscr{H}$. Since $f = g + f^0$ we must have $g = c_1 f$ and $f^0 = c_2 f$ since f is extreme. But $c_2 < 1$ is impossible for then f^0 would not be λ-harmonic. Thus $f = P_t^\lambda f$ for any extreme ray. By linearity and Choquet, $h = P_t^\lambda h$ for any h in $\mathscr{H}$.

So if $\mathscr{H} = \mathscr{H}_\lambda \cap P^t\mathscr{H}_\lambda$ is closed and nontrivial it *consists entirely of complete λ-harmonic functions*.

Example (8.3). If M is the interior of a compact manifold with boundary, a continuous positive λ-harmonic function ϕ is rarely complete. By (4.1) it is necessary that ϕ vanishes on the boundary. Thus $\lambda = \lambda_0$ and ϕ must be proportional to zeroth eigenfunction ϕ_0, which is complete.

Example (8.4) (Another argument). If M (or a covering space) has *bounded geometry*, that is each point is centered in a neighborhood of fixed radius which is a bounded distortion of the unit ball in Euclidean space, then every positive λ-harmonic function is complete. This follows because the constants in Harnack's principle are uniform (so a positive λ-harmonic function ϕ grows at most exponentially) and the heat kernel satisfies an inequality $p_t(x, y) \leqslant ce^{a(d(x,y))^2}$ for $t \leqslant 1$ and $d(x, y) \geqslant 1$ (so $p_t(x, y)\phi(y)\,dy$ has little mass near infinity). Now a straightforward estimate shows that a positive λ-harmonic function is complete.

Problem (8.5) (Stroock and Sullivan). Which open connected manifolds have *complete* positive λ_0-harmonic functions? (See [24].)

We now turn to the proofs of the theorems in §2.3.

9. Proof of Theorems (2.11) and (2.16)

If for some λ there is a positive λ-harmonic function ϕ on $\mathbf{H}^{d+1}$, then we can average ϕ over the compact group of isometries fixing some y in $\mathbf{H}^{d+1}$. We obtain a spherically symmetric positive λ-harmonic function

$\phi(R) = S_\lambda(x, y)$, where $R = d(x, y)$. Then $\phi(R)$ satisfies

$$\left(\frac{d^2}{dR^2} + \frac{A'(R)}{A(R)}\frac{d}{dR} - \lambda\right)\phi = 0, \tag{9.1}$$

where $A(R) =$ the area of the sphere of radius R about y, and $A'(R) = (d/dR)A(R)$.

For R near zero and infinity respectively, this equation becomes

$$\begin{aligned} R = 0&: \quad \left(\frac{d^2}{dR^2} + \frac{d+1}{R}\frac{d}{dR} - \lambda\right)\phi = 0, \\ R = \infty&: \quad \left(\frac{d^2}{dR^2} + d\frac{d}{dR} - \lambda\right)\phi = 0. \end{aligned} \tag{9.2}$$

The exponential solutions near ∞ are determined from the indicial equation $u^2 + du - \lambda = 0$. In other words if $\alpha = -u$, $\lambda = \alpha(\alpha - d)$. Real exponentials result iff $\lambda \geqslant -\frac{1}{4}d^2$. Thus there are spherically symmetric positive λ-harmonic functions iff $\lambda \geqslant -\frac{1}{4}d^2$. This proves Theorem (2.11)(i).

Before proving Theorem (2.11)(ii) we must prove Theorem (2.16) and analyze the λ-Green's function, $g_\lambda(x, y) = \int_0^\infty e^{-\lambda t}p_t(x, y)\,dt$. Looking again at the equations in the form (9.2) one sees:

(i) Near $R = 0$ there is a 1-dimensional subspace of bounded solutions, the rest of the solutions have a standard Green's singularity, $\log(1/R)$ if $d = 1$ and $(1/R)^{d-1}$ if $d > 1$.

(ii) At $R = \infty$ there is a 1-dimensional space of solutions asymptotic to const $\cdot\, e^{-\alpha R}$, where $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$. The rest are asymptotic to const $\cdot\, e^{-(d-\alpha)R}$ if $\alpha > \frac{1}{2}d$ or const $\cdot\, Re^{-(d/2)R}$ if $\alpha = \frac{1}{2}d$.

We know from Theorem (2.11) and Theorem (2.6) and the nonuniqueness of positive λ_0-harmonic functions that the Green's region is $[\lambda_0, \infty)$. We know from $(\Delta_x - \lambda)g_\lambda(x, y) =$ dirac mass at y that $g_\lambda(x, y)$ has a standard Green's singularity at $x = y$, $R = 0$.

We have seen from the definition that $S_\lambda(x, y)$ is bounded near $R = 0$ and therefore $S_\lambda(x, y)$ is the small (or recessive) solution near $R = 0$. We want to show that $g_\lambda(x, y)$ is the small (or recessive) solution at $R = \infty$.

Claim (9.1). The recessive solution at $R = \infty$ for $\lambda \geqslant \lambda_0$ is positive for all $R > 0$ and has a Green's singularity at $R = 0$.

Proof of claim. The bounded solution at $R = 0$, $S_\lambda(x, y)$, has the simple formula

$$\int \phi(x;\, \xi, \alpha)\,d\theta,$$

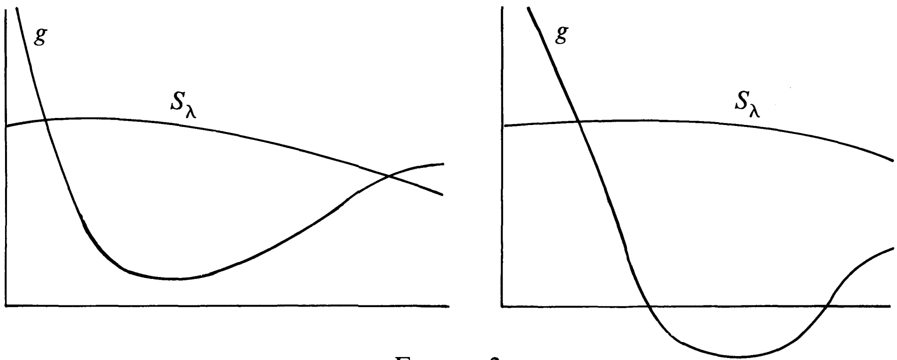

FIGURE 2

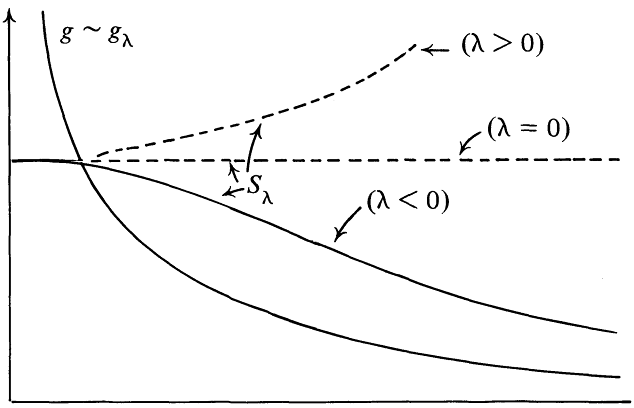

FIGURE 3

where $d\theta$ is the spherical measure on S^d with y the center of the unit ball model and the $\phi(\cdot\,;\, \xi, \alpha)$ of §2.3 are normalized at y.

A special case of the calculation in the proposition of the proof of Theorem (2.13) shows that S_λ is a large solution near $R = \infty$. Thus g, the recessive solution at $R = \infty$, cannot also be recessive at $R = 0$ because it would then be a multiple of S_λ (which is large at $R = \infty$).

Thus g tends to ∞ as $R \to 0$ and must cross S_λ for some smallest $R = R_0$. At R_0 the Wronskian $gS'_\lambda - S_\lambda g' = g(R_0)(S'_\lambda - g')$ is negative since $g(R_0) = S_\lambda(R_0) > 0$, and $S'_\lambda(R_0) < g'(R_0)$. Since the Wronskian does not change sign and $S'_\lambda < 0$, each of the behaviors depicted in Figure 2 is ruled out. So $g > 0$ and we have Figure 3 which proves the claim and a bit more. q.e.d.

To finish the proof that $g = \text{const} \cdot g_\lambda(x, y)$ write $g_\lambda(x, y)$ as the $\sup_\alpha g_\lambda^\alpha(x, y)$, where D_α is an exhaustion of $\mathbf{H}^{d+1}$ by balls centered at y, and

$g_\lambda^\alpha(x, y)$ is the λ-Green's function for D_α. Now $c_1 g - c_2 S_\lambda$ is zero on ∂D_α and has the same weight singularity at $R = 0$, where c_1 and c_2 are positive constants. So $c_1 g - c_2 S_\lambda = g_\lambda^\alpha(x, y)$. Thus $g_\lambda^\alpha(x, y) \leqslant \text{const} \cdot g$. The constant is fixed, so $g_\lambda(x, y) = \sup_\alpha g_\lambda^\alpha(x, y) \leqslant \text{const} \cdot g$. It follows that $g_\lambda(x, y)$ is small (or recessive) at $R = \infty$ and must be a constant times g. This completes the proof of Theorem (2.16).

Now we are in a position to prove Theorem (2.11)(ii) by Martin's construction (1941). We sketch the steps of this famous argument.

Choose a reference point x_0 in $\mathbf{H}^{d+1}$ and consider the quotient $k_\lambda(x, y) = g_\lambda(x, y)/g_\lambda(x_0, y)$. As a function of y (x fixed), $k_\lambda(x, y)$ is continuous on $\mathbf{H}^{d+1} \cup S^d$ with $k_\lambda(x, \xi) = \phi(x, \xi, \alpha)$ (normalized at x_0) for ξ in S^d. This follows from Theorem (2.16), $\alpha = \frac{1}{2}d(\lambda + \frac{1}{4}d^2)^{1/2}$.

Let ϕ be a positive λ-harmonic function which is a limit of λ-potentials

$$\phi_n(x) = \int_y g_\lambda(x, y)\, d\mu_n(y)$$

of Radon-measures μ_n on $\mathbf{H}^{d+1}$ (all are as we shall see). The measures $\mu'_n = g_\lambda(x_0, y)\mu_n$ have total mass $\leqslant \phi_n(x)$ ($\leqslant \phi(x) + 1$ for n large). So let μ be a weak limit measure in $\mathbf{H}^{d+1} \cup S^d$. Since $(\Delta - \lambda)\phi_n = \mu_n$ and $(\Delta - \lambda)\phi = 0$, μ must be supported on S^d. We calculate

$$\begin{aligned}
\phi(x) &= \lim_n \phi_n(x) = \lim_n \int_y g_\lambda(x, y)\, d\mu_n(y) = \lim_n \int_y k_\lambda(x, y)\, d\mu'_n(y) \\
&= \int_y k_\lambda(x, y)\, d\mu \quad \text{(because } k_\lambda(x, y) \text{ is a continuous function of } y\text{)} \\
&= \int_\xi \phi(x, \xi, \alpha)\, d\mu(\xi),
\end{aligned}$$

since μ lives on S^d. This proves the existence part of Theorem (2.11)(ii) for a limit of potentials.

We now give the classical argument to see that any λ-superharmonic function f is an increasing limit of potentials. Form $f_n = \min\{f, nG_\lambda\chi_n\}$, where χ_n is the characteristic function of the ball of radius n about some fixed point and $G_\lambda\chi_n(x) = \int_M \chi_n(y) g_\lambda(x, y)\, dy$. Then f_n is nonnegative bounded, λ-superharmonic, f_n increases to f, and f_n satisfies $\inf_{T \to \infty} P_T^\lambda f_n = 0$ (the latter, since this is true for $nG_\lambda\chi_n$ and $\inf\{P_T^\lambda f, P_T^\lambda g\} \geqslant P_T^\lambda \inf\{f, g\}$).

Now apply (5.1) to f_n and let $T \to \infty$ to obtain

$$G_\lambda\big(1/t\big(f_n - P_t^\lambda f_n\big)\big) = 1/t \int_0^t P_s^\lambda f_n\, ds.$$

The right-hand side is increasing to f_n as $t \to 0$ since f_n is λ-superharmonic. Thus f_n is the increasing limit of potentials $G_\lambda \mu_t$ where $\mu_t = 1/t(f_n - P_t^\lambda f_n)$. This implies that f is the increasing union of potentials and completes the proof of the existence part of Theorem (2.11)(ii).

The uniqueness follows from Theorem (2.12) (which only uses the existence part of Theorem (2.11)(ii) in its proof).

10. Proof of Theorems (2.12), (2.13), (2.14), and (2.15)

To prove Theorem (2.12) we must first calculate the normalizing factor for $\mu(p, \phi, R) = 1/c_R \cdot (\phi/S(p, R)) \cdot$ spherical measure. We want

$$\phi(p) = 1/c_R \int_x \phi/S(p, R)\, d\theta_R(x),$$

where $d\theta_R$ is the unit spherical measure on $S(p, R)$. Write ϕ as an integral of the $\phi(\cdot, \xi, \alpha)$,

$$\phi(x) = \int_\xi \phi(x, \xi, \alpha)\, d\mu(p, y)(\xi),$$

where $\mu(p, \phi)$ has total mass $\phi(p)$. Substituting gives

$$\begin{aligned} \phi(p) c_R &= \int_x \int_\xi \phi(x, \xi, \alpha)\, d\mu(p, \phi)(\xi)\, d\theta_R(x) \\ &= \int_\xi \left(\int_x \phi(x, \xi, \alpha)\, d\theta_R(x) \right) d\mu(p, \phi)(\xi). \end{aligned}$$

Thus c_R is the function of R, $S_\lambda(R) = \int_x \phi(x, \xi, \alpha)\, d\theta_R(x)$ where $x = (R, \xi)$, which we have seen in §9 to be of the order $e^{-(d-\alpha)R}$ for $\alpha > \frac{1}{2}d$ and $Re^{-(d/2)R}$ for $\alpha = \frac{1}{2}d$. With the indicated choice of c_R the total mass of $\mu(p, \phi, R)$ is $\phi(p)$.

Now let f be a continuous function on the compactification of $\mathbf{H}^{d+1}$ by $\mathbf{H}^{d+1} \cup S^d$ and let $R \to \infty$. Then

$$\begin{aligned} \frac{1}{c_R} \int f\, d\mu(p, \phi, R) &= \frac{1}{c_R} \int f \cdot \phi \cdot d\theta_R \\ &= \frac{1}{c_R} \int_x f \left(\int_\xi \phi(x, \xi, \alpha)\, d\mu(p, \phi)(\xi) \right) d\theta_R(x) \\ &= \int_\xi \left(\frac{1}{c_R} \int_x f(x) \cdot \phi(x, \xi, \alpha)\, d\theta_R(x) \right) d\mu(p, \phi)(\xi). \end{aligned}$$

Outside a disk of radius $\varepsilon > 0$ (fixed so that f is near $f(\xi)$ on this region) in polar coordinates (R, ξ), $\phi(x, \xi, \alpha)$ is of the order $e^{-\alpha R}$. On the other hand, the integral $\int \phi(x, \xi, \alpha)\, d\theta_R$ is larger, $e^{-(d-\alpha)R}$ or $Re^{-(d/2)R}$ as indicated above.

Thus the inner integral is concentrated near ξ and converges to $f(\xi)$ on $R \to \infty$. Thus

$$\lim_{R \to r\infty} \int f \cdot d\mu(p, \phi, R) = \int f d\mu(p, \phi),$$

proving Theorem (2.12).

Remark (10.1). This proof of Theorem (2.12) for $\alpha > \frac{1}{2}d$ was shown to me by Mary Rees who offered it as an alternative to the sketch of part (ii) for $\alpha > \frac{1}{2}d$ in [26]. The questions of Mary Rees were part of the motivation for the exposition here.

Now we prove Theorem (2.13). First we have a proposition asserting that no finite measure μ on S^d is more diffuse than Lebesgue measure.

Proposition (10.2). *Let μ be a finite positive measure on S^d. Then for μ-almost all ξ in S^d,*

$$\liminf_{r \to 0} \frac{\mu(\xi, r)}{r^d} > 0,$$

where $\mu(\xi, r)$ is the μ-measure of a disk of radius r centered at ξ.

Proof. Let A be the set of ξ in S^d so that for every $\delta > 0$ and ξ in A there is a sequence $r_i \to 0$ with $\mu(\xi, r_i) \leqslant \varepsilon r_i^d$. By the covering lemma [9, Theorem 2.8.14] there are (arbitrarily fine) coverings of A using disks of these radii (and centers on A) which fall into $K = K(d)$ collections consisting of disjoint disks.

One of these collections C must contain at least $1/K \cdot \mu(A)$ of the mass of μ. Thus

$$1/K \cdot \mu(A) \leqslant \sum_C \mu(\xi, r_i) \leqslant \varepsilon \sum r_i^d$$
$$\leqslant \varepsilon \cdot \text{Lebesgue measure of } S^d.$$

So $\mu(A) \leqslant \varepsilon \cdot K \cdot$ measure of S^d for any $\varepsilon > 0$. This proves the proposition.

Fix ξ_0 and calculate for $x = (R, \xi_0)$

$$\phi(x) = \int_\xi \phi(x, \xi, \alpha)\, d\mu(p, \phi)(\xi).$$

Divide the integral into three parts: (i) $d(\xi, \xi_0) \leqslant e^{-R}$, (ii) $e^{-R} \leqslant d(\xi, \xi_0) \leqslant \varepsilon$, and (iii) $d(\xi, \xi_0) \geqslant \varepsilon$. Here $\varepsilon > 0$ is a parameter and d is the spherical or Euclidean distance in the unit ball model.

An elementary calculation (see [27, §1]) shows that for $x = (\xi_0, R)$ in these three regions $\phi(x, \xi, \alpha)$ is comparable to

$$\text{(i) } e^{+\alpha R}, \qquad \text{(ii) } e^{-\alpha R}/s^{2\alpha}, \qquad \text{(iii) } e^{-\alpha R},$$

where $s = d(\xi, \xi_0)$. Thus

$$\phi(x) = \phi(\xi, R) = \int_{\text{(i)}} e^{\alpha R}\, d\mu + \int_{\text{(ii)}} e^{-\alpha R}/s^{2\alpha}\, d\mu + \int_{\text{(iii)}} e^{-\alpha R}\, d\mu.$$

The first term is comparable to $e^{\alpha R}\mu(\xi_0, e^{-R})$. The third term is at most $e^{-\alpha R}$. We treat the second term by partial integration to obtain (ignoring constants)

$$\begin{aligned} &e^{-\alpha R}\left(\int_{e^{-R} < s \leqslant \varepsilon} \mu(\xi_0, s)/s^{2\alpha+1}\, ds - \mu\left(\xi_0, e^{-R}\right)/e^{-2\alpha R} + C(\varepsilon)\right) \\ &\quad = e^{-\alpha R}\int_{e^{-R} \leqslant s \leqslant \varepsilon} \mu(\xi_0, s)/s^{2\alpha+1}\, ds - (\text{const} \cdot \text{first term}) + \text{const}(\varepsilon). \end{aligned}$$

Now by the previous proposition for μ-almost all ξ_0, $\mu(\xi_0, s)$ is eventually $\geqslant c(\xi_0)s^d$. So, $\mathrm{II} = e^{-\alpha R}\int_{e^{-R} \leqslant s \leqslant \varepsilon} \mu(\xi_0, s)/s^{2\alpha+1}\, ds$ is

$$\geqslant \begin{cases} e^{-(d-\alpha)R} & \text{if } \alpha > \tfrac{1}{2}d, \\ Re^{-(d/2)R} & \text{if } \alpha = \tfrac{1}{2}d. \end{cases} \tag{10.1}$$

It follows that for R large either the first term (i) is at least as large as II or the second term (ii) is of the order of II. Thus (i) + (ii) is at least as large as II which is much bigger than (iii). This proves Theorem (2.13)(b). The others are easier.

We have also derived the fact that the essential contribution to $\phi(x) = \phi(R, \xi_0)$ for R large ($\geqslant R(\varepsilon)$) and μ-almost all ξ_0 comes from the part of the integral with $d(\xi, \xi_0) \leqslant \varepsilon$ for any $\varepsilon > 0$. This is useful for Theorem (2.14).

We now write out

$$\phi_1(x) = \int_\xi \chi(\xi)\phi(x, \xi, \alpha)\, d\mu, \qquad \phi_2(x) = \int_\xi 1 \cdot \phi(x, \xi, \alpha)\, d\mu,$$

where $\mu = \mu(p, \phi_2)$ and $\chi(\xi) = d\mu(p, \phi_1)/d\mu(p, \phi_2)(\xi)$. By the above for μ-almost all ξ_0 and for R large we only need consider the integrals for $d(\xi, \xi_0) < \varepsilon$.

Now consider a set A of ξ of positive μ-measure where $\chi(\xi)$ is approximately a. For $x = (R, \xi_0)$, $\phi(x, \xi, \alpha)$ only depends on $d(\xi, \xi_0)$, as indicated above. Moreover, $\phi(x, \xi, \alpha)$ only varies up to a constant near 1 in ratio on annuli of a definite shape around ξ_0 (again, from the above).

For each ξ_0 in a subset $B \subset A$ of full μ-measure we can choose ε so that if we divide the ε-disk about ξ_0 into concentric annuli of (relative) constancy for $\phi(x, \xi, \alpha)$ $(x = (R, \xi_0),\ R > R(\varepsilon))$ each of these annuli will be mostly filled (relative to μ) by points of A, and the μ-integral of χ on each is approximately a. This follows from Lebesgue density and differentiation. Then we see that $\phi_1(x)$ and $\phi_2(x)$ are sums of terms in approximate ratio a which is approximately $\chi(\xi_0)$. These sets A fill up μ. This proves Theorem (2.14).

Now we turn to the proof of Theorem (2.15). Let A be a set of positive μ-measure so that $\phi(R, \xi) \leqslant e^{(\sigma+\varepsilon)R}$ for $\varepsilon > 0$ and $R > R(\xi, \varepsilon)$. Fixing ε we can make $R(\xi, \varepsilon)$ independent of ξ by reducing A a little to B. Write $\delta = \sigma + \varepsilon$ and $r = e^{-R}$. Referring to the decomposition of the integral for $\phi(x)$ above, we deduce that the first term is $\leqslant e^{\delta R}$. Thus $\mu(\xi_0, r) \leqslant r^{\alpha-\delta}$ for any ξ_0 in B.

For any covering of B by balls of radius r_i centered at ξ_i in B we have

$$0 < \mu(B) \leqslant \sum_i \mu(\xi_i, r_i) \leqslant \sum r_i^{\alpha-\delta}.$$

Thus the Hausdorff $(\alpha - \delta)$-measure of B is positive. So the Hausdorff dimension of $A \supset B$ is $\geqslant \alpha - \delta = \alpha - \sigma + \varepsilon$ for every $\varepsilon > 0$. This proves Theorem (2.15).

11. Proof of Theorems (2.17), (2.19), (2.21), (2.23), and (2.24)

If $M = \mathbf{H}^{d+1}/\Gamma$, then $p_t^M(x, y)$ is just $\sum_{\gamma \in \Gamma} p_t(x^0, \gamma y^0)$, where x^0, y^0 lie in $\mathbf{H}^{d+1}$ over x, y. Thus $g_\lambda^M(x, y) = \sum_{\gamma \in \Gamma} g_\lambda(x^0, \gamma y^0)$. So if x^0 is not on the Γ orbit of y^0, then $g_\lambda^M(x, y)$ has the order of the Poincaré series $\sum_\Gamma \exp(-\alpha d(x^0, \gamma y^0))$ by Theorem (2.16), $\alpha = \frac{1}{2}d + (\lambda + \frac{1}{4}d^2)^{1/2}$. Thus $g_\lambda^M(x, y) < \infty$ for $x \neq y$ if $\alpha > \delta(\Gamma)$ and $g_\lambda^M(x, y) = \infty$ for $\alpha < \delta(\Gamma)$ when $\delta(\Gamma) \geqslant \frac{1}{2}d$. This means $\lambda_0(M) = \delta(\Gamma)(\delta(\Gamma) - d)$ if $\delta(\Gamma) \geqslant \frac{1}{2}d$ by Theorem (2.6). Otherwise $\lambda_0(M) = -\frac{1}{4}d^2$, since $\lambda_0(M) \geqslant -\frac{1}{4}d^2$ by Theorem (2.1) and Theorem (2.11). This proves Theorem (2.17).

Theorem (2.19) is partially proved in [26] generalizing [20], namely $\delta(\Gamma)$ is the minimum power (which is achieved) for a measure satisfying (2.1) [26, §2].

If $\delta > \delta(\Gamma)$, put a Dirac mass at each point of the orbit $\gamma(y)$ of a point y in the open ball model B^{d+1} of $\mathbf{H}^{d+1}$ with weight $|\gamma' y|^\delta$. A measure of finite mass results because the Poincaré series converges at $\delta > \delta(\Gamma)$. This measure satisfies (2.1) but is not supported on S^d. The set of measures of bounded mass satisfying (10.1) supported in the closed ball is a closed set. Thus let y approach infinity in a fundamental domain and take a limit to prove Theorem (2.19)(i).

To prove Theorem (2.19)(ii) we merely let y approach a limit point staying in one fundamental domain (then all the mass approaches the limit set) and this is possible unless Γ is geometrically finite without cusps.

In that case there is only a measure of exponent $\delta(\Gamma)$, i.e. (2.1) for $\delta = \delta(\Gamma)$, and this is Hausdorff measure by [26, §3]. This completes the proof of Theorem (2.19).

To prove Theorem (2.21)(i) we merely quote [28], which proves $\delta(\Gamma) =$ the Hausdorff dimension of the limit set for geometrically finite groups, and apply Theorem (2.17). Part (ii) also follows from [28]. Thus Theorem (2.21) is proved. The corollary is a local consequence.

Theorem (2.23) follows from Theorem (2.17) and [26, Corollary 27]. Theorem (2.24) follows from Theorem (2.21) and the definitions (see [14]).

Acknowledgements. During the work on this paper I have benefited from remarks, questions, and discussions with Phil Trauber, Mary Rees, Jon Aaronson, Henry McKean, Ragu Varadhan, Peter Lax, Dan Stroock, Dick Holley, Robert Brooks, and Bert Kostant, in chronological order.

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