ON THE INTERSECTION RING OF COMPACT THREE MANIFOLDS

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If \( M \) is a compact oriented manifold of dimension three we have the following invariant defined by intersection theory—take three oriented surfaces \( x, y, \) and \( z \) in \( M, \) form their transversal intersection and calculate the algebraic number of intersection points. This defines a skew symmetric three form \( \mu_M \) on the space \( H = H'(M) \sim H_3(M). \)

\( \mu_M \) completely determines the intersection ring (mod torsion) of \( M \) because of Poincaré duality. On the other hand the space of possible tensors \( \mu \) on a given space \( H \) up to the natural equivalence is quite complex in general. If \( \beta = \text{dimension} \ H, \mu \) is specified by \( \frac{1}{2} \beta(\beta - 1)(\beta - 2) \) parameters with an ambiguity of the action of the group \( \text{Gl}(H) \) of dimension \( \beta^2. \)

We will point out three facts concerning the algebraic invariant \( \mu \) and the topology of three-manifolds.

(i) Given any free abelian group \( H \) of finite rank and any skew symmetric three form \( \mu \) on \( H \) there is 3-manifold \( M \) so that \( H'(M) = H \) and \( \mu_M = \mu. \)

(ii) If \( V \) is a complex algebraic surface with an isolated singularity at \( p \in V, \) then the link of \( p \) is a compact oriented three manifold \( L_p \) and \( \mu(L_p) = 0. \) For example this shows \( S^1 \times S^1 \times S^1 \) is not a link.

(iii) If \( \pi \) is the fundamental group of \( M', \) then \( \pi \) made abelian (mod torsion) is isomorphic to the dual space of \( H \) and the commutator construction \( (a, b) \rightarrow [a, b] = aba^{-1}b^{-1} \) defines a skew symmetric pairing.

\[
\pi/[\pi, \pi] \otimes \pi/[\pi, \pi] \xrightarrow{1,1} \{\pi, \pi]/[\pi, [\pi, \pi]].
\]

If we dualize this \([,] \) pairing we find the degeneracy of \( \mu \) or dually the relations among commutators namely an exact sequence (mod torsion)

\[
0 \rightarrow ([\pi, \pi]/[\pi[\pi, \pi]])^* \xrightarrow{(1, \eta)} \Lambda^2 H \xrightarrow{\mu} H^*.
\]

For example, if \( \mu = 0, \pi_*M \) looks like the free group on \( \beta = \text{dim} \ H \) generators (mod torsion and triple commutators). In general there are at most \( \beta \) relations (defined by \( \mu \)) among the \( \frac{1}{2} \beta(\beta - 1) \) commutators.

The Proofs

The proof of (i) is the most interesting. We want to construct a 3 manifold \( M \) with \( H'(M) \sim H \) and \( \mu_M = \mu \) with \( (H, \mu) \) arbitrary. Let \( \beta = \text{dim} \ H \) and consider the solid torus \( S \) of genus \( \beta \) and boundary \( \partial S, \) the surface of genus \( \beta. \)

We identify \( H \) with the kernel of the inclusion map \( H, \partial S \rightarrow H, S \) and let \( G \) be the subgroup of homeomorphisms \( \sigma \) of \( \partial S \) satisfying \( \sigma/H = \text{identity}. \)

We can form for each such homeomorphism \( \sigma \) the three manifold \( M_\sigma = S \cup \sigma S \) by pasting two copies of \( S \) together along \( \partial S \) using \( \sigma. \) It is easy to check that \( H, M_\sigma \) is isomorphic to \( H \) but is not quite so easy to identify the \( \mu \) invariant, \( \mu(M_\sigma) \) which will turnout to be the general \( \mu. \)

Now we have the following additivity proposition:

if \( \sigma, \sigma' \in G, \) then

\[
\mu(M_{\sigma \circ \sigma'}) = \mu(M_\sigma) + \mu(M_{\sigma'}).
\]
Granting this for the moment the proof proceeds by analyzing a Heegard decomposition of
three torus $T = S^1 \times S^1 \times S^1$. We will write $T = M_\sigma$ for $\sigma \in G$ and since $\mu(T) = x_1 \wedge x_2 \wedge x_3$, we
will be done by additivity. Consider $T$ as the quotient of euclidean space $\mathbb{R}^3$ by the integral
translations in three orthogonal directions with the standard cube in $R^3$ as the fundamental
domain. Consider the two systems of lines $L_1$ and $L_2$ obtained by translating the edges of the
basic cube on the one hand and the perpendicular axes of symmetry through the faces of the cube
on the other hand. If we carefully thicken $L_1$ and $L_2$ to tubular neighborhoods we obtain an
infinite equivariant Heegard decomposition of $\mathbb{R}^3$ which yields upon identification a
decomposition, $S^1 \times S^1 \times S^1 = S \cup \_ S$, where $\delta S$ has genus 3 and $\sigma: \delta S \to \delta S$ induces the identity
on $H = \text{kernel} \ H_1 \delta S \to H_2 S$.

One calculates visually that $\sigma$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
$$

in the natural basis, after reflection (see Fig. 1).

Now consider the general case of $H$ with basis $x_1, \ldots, x_6$ and $\mu = \sum_{i<j<k} a_{ijk} x_i \wedge x_j \wedge x_k$ in $\Lambda^3 H$,
$a_{ijk}$ integers. Let $S_\beta$ denote the solid torus with $\beta$ one handles $h_1, \ldots, h_\beta$ attached. For a moment
think of $h_1, h_2$, and $h_3$ attached to the upper hemisphere of the boundary of the ball and the others
to the lower. Construct $\sigma_{ijk}$, a homeomorphism of $\delta S_\beta$ using the $\sigma$ considered before in the
decomposition of $S^1 \times S^1 \times S^1$ on the upper part of $\delta S_\beta$ and the identity on the lower part. It is
clear that $M_{\sigma_{ijk}} \equiv (S^1 \times S^1 \times S^1)$ connected sum $(\beta - 3 \text{ copies of } S^1 \times S^2)$, so $\mu(M_{\sigma_{ijk}}) = x_i \wedge x_j \wedge x_k$.

Now since $\sigma_{ijk}$ is in the subgroup fixing kernel $H_1 \delta S_\beta \to H_2 S_\beta$, we can calculate $\mu(M_{\sigma_{ijk}})$ where

$$
\tau = \prod_{i<j<k} (\sigma_{ijk})^{a_{ijk}}
$$

by the additivity proposition to be the desired

$$
\mu = \sum_{i<j<k} a_{ijk} x_i \wedge x_j \wedge x_k.
$$

Note that by varying the order of the $\sigma_{ijk}$'s in $\tau$ we obtain many different manifolds with
invariant $\mu$. Each one however has no torsion in $H_1$ and its genus (the minimal genus in a Heegard
decomposition) equals the first Betti number $\beta$.

The additivity proposition can be proved directly using homologies in a collar neighborhood
of $\delta S$ between $\{x_i\}$ the standard basis of $H$ and $\{u(x_i)\}$ on the cycle level. The additivity follows
because we can calculate independently in separate collars for $\sigma$ and $\sigma'$ to obtain the sum $\mu$ for $\sigma \cdot \sigma'$. For this very simple and elegant proof of the additivity the author is indebted to François
Laudenbach (see Fig. 2).

The proof of statement (ii) above follows immediately from the existence of a four manifold

Fig. 1.
W* so that \( \partial W^* \) = link of singularity \( L \) and the intersection pairing on \( H_2 W^* \) is non degenerate (in fact the resolution theory used to produce \( W^* \) shows the pairing is negative definite).

Whenever a three manifold bounds such a \( W^* \) we see that \( \mu \) vanishes by considering the exact sequence with rational coefficients,

\[
H^1 W \rightarrow H^1 \partial W \rightarrow H^2(W, \partial W) \rightarrow H^2 W \rightarrow H^2 \partial W.
\]

The hypothesis on \( W \) means \( i \) is an isomorphism, thus \( t \) and \( t' \) are zero. Since \( t \) is zero \( r \) is onto and cup products in \( \partial W \) can be calculated by lifting to \( W \), multiplying, and restricting by \( t' \). Thus \( \mu = 0 \).

The proof of (iii) is a certain amount of soul searching classical algebraic topology.

This study was motivated by the more general question of algebraic invariants of manifolds and their homeomorphism type. In higher dimensions (assuming simple connectivity) the intersection ring plays an important role in the structure and all such rings (over \( \mathbb{Q} \)) satisfying duality occur geometrically if the Thom–Hirzebruch Index Theorem is valid. This is proved by surgery and the theory in [1] which also led to the additivity proposition above and helps one understand remarks like (iii).

REFERENCE


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