Topology Vol. 14, pp. 275–277. Pergamon Press 1975. Printed in Great Britain

ON THE INTERSECTION RING OF COMPACT THREE MANIFOLDS

DENNIS SULLIVAN

(*Received* 18 *May* 1974)

IF M is a compact oriented manifold of dimension three we have the following invariant defined by intersection theory—take three oriented surfaces x, y, and z in M, form their transversal intersection and calculate the algebraic number of intersection points. This defines a skew symmetric three form μ_M on the space $H = H^1(M) \simeq H_2M$.

μ_M completely determines the intersection ring (mod torsion) of M because of Poincaré duality. On the other hand the space of possible tensors μ on a given space H up to the natural equivalence is quite complex in general. If $\beta =$ dimension H, μ is specified by $\frac{1}{6}\beta(\beta-1)(\beta-2)$ parameters with an ambiguity of the action of the group $Gl(H)$ of dimension β^2.

We will point out three facts concerning the algebraic invariant μ and the topology of three-manifolds.

(i) Given any free abelian group H of finite rank and any skew symmetric three form μ on H there is 3-manifold M so that $H^1(M) \simeq H$ and $\mu_M \simeq \mu$.

(ii) If V is a complex algebraic surface with an isolated singularity at $p \in V$, then the link of p is a compact oriented three manifold L_p and $\mu(L_p) = 0$. For example this shows $S^1 \times S^1 \times S^1$ is not a link.

(iii) If π is the fundamental group of M^3, then π made abelian (mod torsion) is isomorphic to the dual space of H and the commutator construction $(a, b) \to [a, b] = aba^{-1}b^{-1}$ defines a skew symmetric pairing,

$$\pi/[\pi,\pi] \otimes \pi/[\pi,\pi] \xrightarrow{[,]} [\pi,\pi]/[\pi,[\pi,\pi]].$$

If we dualize this [,] pairing we find the degeneracy of μ or dually the relations among commutators namely an exact sequence (mod torsion)

$$0 \to ([\pi,\pi]/[\pi[\pi,\pi]])^* \xrightarrow{[,]^*} \Lambda^2 H \xrightarrow{\mu} H^*.$$

For example, if $\mu = 0$, $\pi_1 M$ looks like the free group on $\beta = \dim H$ generators (mod torsion and triple commutators). In general there are at most β relations (defined by μ) among the $\frac{1}{2}\beta(\beta-1)$ commutators.

The Proofs

The proof of (i) is the most interesting. We want to construct a 3-manifold M with $H^1M \simeq H$ and $\mu_M \simeq \mu$ with (H, μ) arbitrary. Let $\beta = \dim H$ and consider the solid torus S of genus β and boundary ∂S, the surface of genus β.

We identify H with the kernel of the inclusion map $H_1\partial S \to H_1 S$ and let G be the subgroup of homeomorphisms σ of ∂S satisfying $\sigma/H =$ identity.

We can form for each such homeomorphism σ the three manifold $M_\sigma = S \cup_\sigma S$ by pasting two copies of S together along ∂S using σ. It is easy to check that $H_1 M_\sigma$ is isomorphic to H but is not quite so easy to identify the μ invariant, $\mu(M_0)$ which will turnout to be the general μ. Now we have the following additivity proposition:

if $\sigma, \sigma' \in G$, then

$$\mu(M_{\sigma\cdot\sigma'}) = \mu(M_\sigma) + \mu(M_{\sigma'}).$$

Granting this for the moment the proof proceeds by analyzing a Heegard decomposition of three torus $T = S^1 \times S^1 \times S^1$. We will write $T = M_\sigma$ for $\sigma \in G$ and since $\mu(T) = x_1 \wedge x_2 \wedge x_3$ we will be done by additivity. Consider T as the quotient of euclidean space R^3 by the integral translations in three orthogonal directions with the standard cube in R^3 as the fundamental domain. Consider the two systems of lines L_1 and L_2 obtained by translating the edges of the basic cube on the one hand and the perpendicular axes of symmetry through the faces of the cube on the other hand. If we carefully thicken L_1 and L_2 to tubular neighborhoods we obtain an infinite equivariant Heegard decomposition of R^3 which yields upon identification a decomposition, $S^1 \times S^1 \times S^1 = S \cup_\sigma S$, where ∂S has genus 3 and $\sigma: \partial S \to \partial S$ induces the identity on $H = \text{kernel } H_1 \ \partial S \to H_1 S$.

One calculates visually that σ is given by the matrix

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ \hdashline & & & 1 & 0 & 0 \\ & \bigcirc & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right)$$

in the natural basis, after reflection (see Fig. 1).

Now consider the general case of H with basis $x_1, \ldots, x_\beta$ and $\mu = \sum_{i<J<k} a_{ijk} x_i \wedge x_j \wedge x_k$ in $\Lambda^3 H$, a_{ijk} integers. Let S_β denote the solid torus with β one handles $h_1, \ldots, h_\beta$ attached. For a moment think of h_i, h_j, and h_k attached to the upper hemisphere of the boundary of the ball and the others to the lower. Construct σ_{ijk}, a homeomorphism of ∂S_β using the σ considered before in the decomposition of $S^1 \times S^1 \times S^1$ on the upper part of ∂S_β and the identity on the lower part. It is clear that $M_{\sigma_{ijk}} \cong (S^1 \times S^1 \times S^1)$ connected sum ($\beta - 3$ copies of $S^1 \times S^2$), so $\mu(M_{\sigma_{ijk}}) = x_i \wedge x_j \wedge x_k$.

Now since σ_{ijk} is in the subgroup fixing kernel $H_1 \partial S_\beta \to H_1 S_\beta$, we can calculate $\mu(M_\tau)$ where

$$\tau = \prod_{i<j<k} (\sigma_{ijk})^{a_{ijk}}$$

by the additivity proposition to be the desired

$$\mu = \sum_{i<j<k} a_{ijk} x_i \wedge x_j \wedge x_k.$$

Note that by varying the order of the σ_{ijk}'s in τ we obtain many different manifolds with invariant μ. Each one however has no torsion in H_1 and its genus (the minimal genus in a Heegard decomposition) equals the first Betti number β.

The additivity proposition can be proved directly using homologies in a collar neighborhood of ∂S between $\{x_i\}$ the standard basis of H and $\{\sigma(x_i)\}$ on the cycle level. The additivity follows because we can calculate independently in separate collars for σ and σ' to obtain the sum μ for $\sigma . \sigma'$. For this very simple and elegant proof of the additivity the author is indebted to François Laudenbach (see Fig. 2).

The proof of statement (ii) above follows immediately from the existence of a four manifold

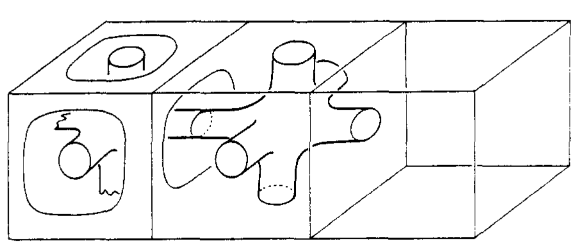

Fig. 1.

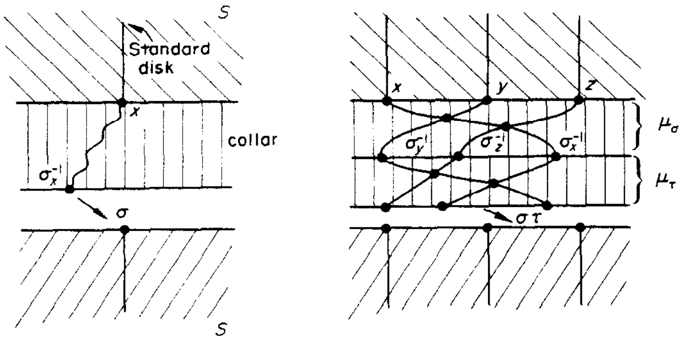

Fig. 2.

W^4 so that ∂W^4 = link of singularity L_p and the intersection pairing on $H_2 W^4$ is non degenerate (in fact the resolution theory used to produce W^4 shows the pairing is negative definite).

Whenever a three manifold bounds such a W^4 we see that μ vanishes by considering the exact sequence with rational coefficients,

$$H^1 W \xrightarrow{r} H^1 \partial W \xrightarrow{t} H^2(W, \partial W) \xrightarrow{i} H^2 W \xrightarrow{t'} H^2 \partial W.$$

The hypothesis on W means i is an isomorphism, thus t and t' are zero. Since t is zero r is onto and cup products in ∂W can be calculated by lifting to W, multiplying, and restricting by t'. Thus $\mu = 0$.

The proof of (iii) is a certain amount of soul searching classical algebraic topology.

This study was motivated by the more general question of algebraic invariants of manifolds and their homeomorphism type. In higher dimensions (assuming simple connectivity) the intersection ring plays an important role in the structure and all such rings (over Q) satisfying duality occur geometrically if the Thom–Hirzebruch Index Theorem is valid. This is proved by surgery and the theory in [1] which also led to the additivity proposition above and helps one understand remarks like (iii).

REFERENCE

1. D. Sullivan: Differential forms and the topology of manifolds, *Tokyo Conference on Manifolds* (1973).

Institute des Hautes Etudes Scientifiques