Parametrized Surfaces

In these notes, we look more closely at surfaces in $\mathbb{R}^n$. As we discussed in section 3.8, a surface $S$ is often given in the form $F(x) = 0$, where $F \in C^1$, but we must be careful here. First of all, this is only assumed to be true locally, so that if $p$ is on $S$, there is an open neighborhood $U$ of $p$ and a $C^1$ function $F : U \to \mathbb{R}$, such that $S \cap U = \{x \in U : F(x) = 0\}$. Secondly, in section 3.8 we discussed the significance of the condition $\partial F/\partial x_n \neq 0$ at a point on the surface; it enables us to locally solve for this $x_n$ on $S$, in terms of the other $x_k$. We will need to demand that this condition be true at each point for some $m$, in other words, that at each point of $S \cap U$, $\text{grad} F \neq 0$. With this condition, since we can locally write $x_n = u(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$ for some $C^1$ function $u$, $S$ is, locally, the graph of a $C^1$ function. Without this condition, many kinds of pathologies can arise. For example, if $F(x_1, x_2) = x_1 x_2$, then $F$ does not satisfy this condition at $(0, 0)$, and $\{x : F(x) = 0\}$ is a pair of straight lines intersecting at 0.

A local parametrization of $S$ is a $C^1$ map $g : V \to \mathbb{R}^n$, where $V \subseteq \mathbb{R}^{n-1}$ is open, and where for some open set $U \subseteq \mathbb{R}^n$, $g(V) = S \cap U$, and where, for each point in $V$, the rank of $g'$ is $n-1$. That is, we demand that the columns of the matrix of $g'$ be linearly independent at each point of $V$. Thus, we would not consider $(x_1, x_2) = (s^3, s^3) (s \in \mathbb{R})$ to be an acceptable parametrization of the surface $\{x : x_2 = x_1\}$ in $\mathbb{R}^2$. The reason for this requirement is that we would like the conclusion of Exercise 0.1 below to hold.

Say now for instance that $\partial F/\partial x_n \neq 0$ at a point in $S$. Then, near that point, by the implicit function theorem, we can parametrize $S$ in the form $(x_1, \ldots, x_{n-1}, x_n) = (s_1, \ldots, s_{n-1}, u(s_1, \ldots, s_{n-1}))$, and this is clearly an acceptable local parametrization. Therefore, local parametrizations will always exist; but this one may not be the most desirable or convenient. For example, in $\mathbb{R}^2$, one could parametrize the unit circle $(x_1)^2 + (x_2)^2 = 1$, for $x_2 > 0$, as $(x_1, x_2) = (s_1, \sqrt{1 - (s_1)^2})$, for $-1 < s_1 < 1$. But it is usually more convenient to parametrize it as $(x_1, x_2) = g(\theta) = (\cos \theta, \sin \theta)$.

Note that in the latter situation, $g'(\theta) = (-\sin \theta, \cos \theta)$ is always tangent to the circle. In general one has:

*Exercise 0.1 Suppose that a surface $S$ is locally given in the form $F(x) = 0$, where $F \in C^1$. In other words, there is an open neighborhood $U$ of $p$ and a $C^1$ function $F : U \to \mathbb{R}$, such that $S \cap U = \{x \in U : F(x) = 0\}$. Suppose that at each point of $S \cap U$, $\text{grad} F \neq 0$. Suppose that $S \cap U$ is also given parametrically as $\{g(s) : s \in V\}$, for a $C^1$ function $g : V \to \mathbb{R}^n$ (here $V \subseteq \mathbb{R}^{n-1}$ is open). Suppose $p = g(s_0) \in S \cap U$. Show that the tangent plane to $S$ at $p$ is the affine space $p + V_1$, where $V_1$ is the span of the columns of $g'(s_0)$. (Of course, we are assuming those columns are linearly independent).
Solution} First we need to verify that $\text{grad} F(p)$ is normal to the tangent plane at $p$. We know that some partial $F_j := \partial F/\partial x_j$ does not vanish at $p$; for each in notation, assume $F_n = \partial F/\partial x_n(p) \neq 0$. The implicit function theorem tells us that, for some neighborhood $W$ of $p$ in $\mathbb{R}^n$, and some neighborhood $W_0$ of $p' := (p_1, \ldots, p_{n-1})$ in $\mathbb{R}^{n-1}$, there is a $C^1$ function $f : W_0 \to W$ such that

$$S \cap W = \{ x \in W : x_n = f(x_1, \ldots, x_{n-1}) \}.$$  

The tangent plane at $p$ therefore has the form

$$x_n - p_n = \sum_{j=1}^{n-1} a_j (x_j - p_j)$$

where each

$$a_j = (\partial f/\partial x_j)(p').$$

Thus the vector $(-a_1, \ldots, -a_{n-1}, 1)$ lies in the normal direction to the plane. Now, on $W_0$, we have

$$F(x_1, x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \equiv 0.$$  

Differentiating this equation with respect to $x_j$ ($1 \leq j \leq n - 1$), and setting $(x_1, \ldots, x_{n-1}) = (p_1, \ldots, p_{n-1})$, we find

$$F_j(p) + F_n(p)a_j = 0.$$  

Accordingly

$$\text{grad} F(p) = F_n(p)(F_1(p)/F_n(p), \ldots, F_{n-1}(p)/F_n(p), 1) = (-a_1, \ldots, -a_{n-1}, 1)$$

is normal to the tangent plane at $p$, as claimed.

Now, on $V$, $F(g(s)) \equiv 0$. Using the chain rule to differentiate this, and setting $s = s_0$, we see that $F'(p)g'(s_0) = 0$. Of course $F'(p)$ is the $1 \times n$ matrix $(F_1(p) \ldots F_n(p))$, so we now see that the columns of $g'(s_0)$ are orthogonal to the vector $n := \text{grad} F(p)$. Since that vector is orthogonal to the tangent plane to $S$ at $p$, that tangent plane has the equation $n \cdot (x - p) = 0$. In other words, the tangent plane to $S$ is the affine space $p + V_2$, where $V_2$ is the set of all vectors $v \in \mathbb{R}^n$ which are orthogonal to $n$. Then surely $V_1 \subseteq V_2$; but by the linear independence assumption, $V_1$ is $n - 1$ dimensional. $V_2$ cannot be $n$-dimensional, so $V_2 = V_1$, and we are done.