Parametrized Surfaces

In these notes, we look more closely at surfaces in $\mathbf{R}^n$. As we discussed in section 3.8, a surface S is often given in the form $F(x) = 0$, where $F \in C^1$, but we must be careful here. First of all, this is only assumed to be true locally, so that if p is on S, there is an open neighborhood U of p and a C^1 function $F : U \to \mathbf{R}$, such that $S \cap U = \{x \in U : F(x) = 0\}$. Secondly, in section 3.8 we discussed the significance of the condition $\partial F/\partial x_m \neq 0$ at a point on the surface; it enables us to locally solve for this x_m on S, in terms of the other x_k. We will need to demand that this condition be true at each point for *some* m, in other words, that at each point of $S \cap U$, grad$F \neq 0$. With this condition, since we can locally write $x_m = u(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$ for some C^1 function u, S is, locally, *the graph of a C^1 function.* Without this condition, many kinds of pathologies can arise. For example, if $F(x_1, x_2) = x_1 x_2$, then F does not saisfy this condition at $(0, 0)$, and $\{x : F(x) = 0\}$ is a pair of straight lines intersecting at 0.

A *local parametrization* of S is a C^1 map $g : V \to \mathbf{R}^n$, where $V \subseteq \mathbf{R}^{n-1}$ is open, and where for some open set $U \subseteq \mathbf{R}^n$, $g(V) = S \cap U$, and where, for each point in V, the rank of g' is $n-1$. That is, we demand that the columns of the matrix of g' be linearly independent at each point of V. Thus, we would not consider $(x_1, x_2) = (s^3, s^3)$ $(s \in \mathbf{R})$ to be an acceptable parametrization of the surface $\{x : x_2 = x_1\}$ in $\mathbf{R}^2$. The reason for this requirement is that we would like the conclusion of Exercise 0.1 below to hold.

Say now for instance that $\partial F/\partial x_n \neq 0$ at a point in S. Then, near that point, by the implicit function theorem, we can parametrize S in the form $(x_1, \ldots, x_{n-1}, x_n) = (s_1, \ldots, s_{n-1}, u(s_1, \ldots, s_{n-1}))$, and this is clearly an acceptable local parametrization. Therefore, local parametrizations will always exist; but this one may not be the most desirable or convenient. For example, in $\mathbf{R}^2$, one could parametrize the unit circle $(x_1)^2 + (x_2)^2 = 1$, for $x_2 > 0$, as $(x_1, x_2) = (s_1, \sqrt{1 - (s_1)^2})$, for $-1 < s_1 < 1$. But it is usually more convenient to parametrize it as $(x_1, x_2) = g(\theta) = (\cos\theta, \sin\theta)$.

Note that in the latter situation, $g'(\theta) = (-\sin\theta, \cos\theta)$ is always tangent to the circle. In general one has:

***Exercise 0.1** *Suppose that a surface S is locally given in the form $F(x) = 0$, where $F \in C^1$. In other words, there is an open neighborhood U of p and a C^1 function $F : U \to \mathbf{R}$, such that $S \cap U = \{x \in U : F(x) = 0\}$. Suppose that at each point of $S \cap U$, grad$F \neq 0$. Suppose that $S \cap U$ is also given parametrically as $\{g(s) : s \in V\}$, for a C^1 function $g : V \to \mathbf{R}^n$ (here $V \subseteq \mathbf{R}^{n-1}$ is open). Suppose $p = g(s_0) \in S \cap U$. Show that the tangent plane to S at p is the affine space $p + V_1$, where V_1 is the span of the columns of $g'(s_0)$. (Of course, we are assuming those columns are linearly independent).*

Solution First we need to verify that $\mathrm{grad} F(p)$ is normal to the tangent plane at p. We know that some partial $F_j := \partial F/\partial x_j$ does not vanish at p; for each in notation, assume $F_n = \partial F/\partial x_n(p) \neq 0$. The implicit function theorem tells us that, for some neighborhood W of p in $\mathbf{R}^n$, and some neighborhood W_0 of $p' := (p_1, \ldots, p_{n-1})$ in $\mathbf{R}^{n-1}$, there is a C^1 function $f : W_0 \to W$ such that

$$S \cap W = \{x \in W : x_n = f(x_1, \ldots, x_{n-1})\}.$$

The tangent plane at p therefore has the form

$$x_n - p_n = \sum_{j=1}^{n-1} a_j (x_j - p_j)$$

where each

$$a_j = (\partial f/\partial x_j)(p').$$

Thus the vector $(-a_1, \ldots, -a_{n-1}, 1)$ lies in the normal direction to the plane. Now, on W_0, we have

$$F(x_1, x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \equiv 0.$$

Differentiating this equation with respect to x_j $(1 \leq j \leq n-1)$, and setting $(x_1, \ldots, x_{n-1}) = (p_1, \ldots, p_{n-1})$, we find

$$F_j(p) + F_n(p) a_j = 0.$$

Accordingly

$$\mathrm{grad} F(p) = F_n(p)\left(\frac{F_1(p)}{F_n(p)}, \ldots, \frac{F_{n-1}(p)}{F_n(p)}, 1\right) = (-a_1, \ldots, -a_{n-1}, 1)$$

is normal to the tangent plane at p, as claimed.

Now, on V, $F(g(s)) \equiv 0$. Using the chain rule to differentiate this, and setting $s = s_0$, we see that $F'(p) g'(s_0) = 0$. Of course $F'(p)$ is the $1 \times n$ matrix $(F_1(p) \ldots F_n(p))$, so we now see that the columns of $g'(s_0)$ are orthogonal to the vector $n := \mathrm{grad} F(p)$. Since that vector is orthogonal to the tangent plane to S at p, that tangent plane has the equation $n \cdot (x - p) = 0$. In other words, the tangent plane to S is the affine space $p + V_2$, where V_2 is the set of all vectors $v \in \mathbf{R}^n$ which are orthogonal to n. Then surely $V_1 \subseteq V_2$; but by the linear independence assumption, V_1 is $n-1$ dimensional. V_2 cannot be n-dimensional, so $V_2 = V_1$, and we are done.