Conformal Welding of Dendrites

Peter Lin

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Reading Committee:

Steffen Rohde, Chair

Donald Marshall

Tatiana Toro

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Abstract

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Peter Lin

Chair of the Supervisory Committee:
Professor Steffen Rohde
Department of Mathematics

We investigate the conformal welding problem, which is a way of taking quotients of Riemann surfaces by identifying points on their boundaries. The existence and uniqueness of this operation is in general difficult to determine. Our focus is on weldings which exhibit branching so that the resulting boundary interfaces are dendrites. We show that the welding relation associated to certain Julia sets in complex dynamics satisfies a regularity condition analogous to the classical quasisymmetry condition. We also show that the Brownian lamination, a random welding relation related to the continuum random tree, has a unique solution.

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Finally, I thank my mother and father, for their love and support.

Chapter 1

INTRODUCTION

Let D_1 and D_2 be topological spaces, and let $\sim$ be an equivalence relation on the disjoint union $D_1 \sqcup D_2$. It is an elementary fact of point set topology that there is a unique (modulo homeomorphism) *quotient space* $Q := D_1 \sqcup D_2 / \sim$ together with a *quotient map* $\pi : D_1 \sqcup D_2 \to Q$ with the following universal property.

If $f_1 : D_1 \to Y$ and $f_2 : D_2 \to Y$ are continuous maps into any other topological space Y, and if f_1 and f_2 are consistent with $\sim$ in the sense that $x \sim y \implies f_1(x) = f_2(y)$, then there is a unique continuous map $\tilde{f} : Q \to Y$ such that $\tilde{f} \circ \pi = f_1 \sqcup f_2$.

Informally speaking, we can say that Q is the result of gluing D_1 and D_2 via $\sim$, and this construction is fundamental in all of mathematics.

This thesis is concerned with an analogous problem in the category of Riemann surfaces, the so called 'conformal welding problem' [31]. In this setting the existence of the 'quotient' is in general not guaranteed to exist, and even when it does, it is not necessarily unique.

A common special case is when the $D_1 = \overline{\mathbb{D}}$ and $D_2 = \overline{\mathbb{D}^*} := \mathbb{C} \backslash \mathbb{D}$ are the closed unit disk and the exterior of the unit disk respectively. Suppose $\gamma \subset \hat{\mathbb{C}}$ is a Jordan curve and let ϕ_-, ϕ_+ be conformal maps from $\mathbb{D}, \mathbb{D}^*$ onto the bounded and unbounded complementary components of $\hat{\mathbb{C}} \backslash \gamma$ respectively. By Carathéodory's theorem, ϕ_- and ϕ_+ extend continuously to their common boundaries $\partial\mathbb{D}$. The *welding homeomorphism*[1] associated to γ is the homeomorphism $h := \phi_+^{-1} \circ \phi_- : \partial\mathbb{D} \to \partial\mathbb{D}$. See Figure 1.1.

[1]Note that we have abused language here because h is only defined up to pre and post composition by Möbius transformations of the circle (since pre-composing ϕ_- or ϕ_+ by a Möbius transformation of the disk yields another conformal map onto a component of $\hat{\mathbb{C}} \backslash \gamma$).

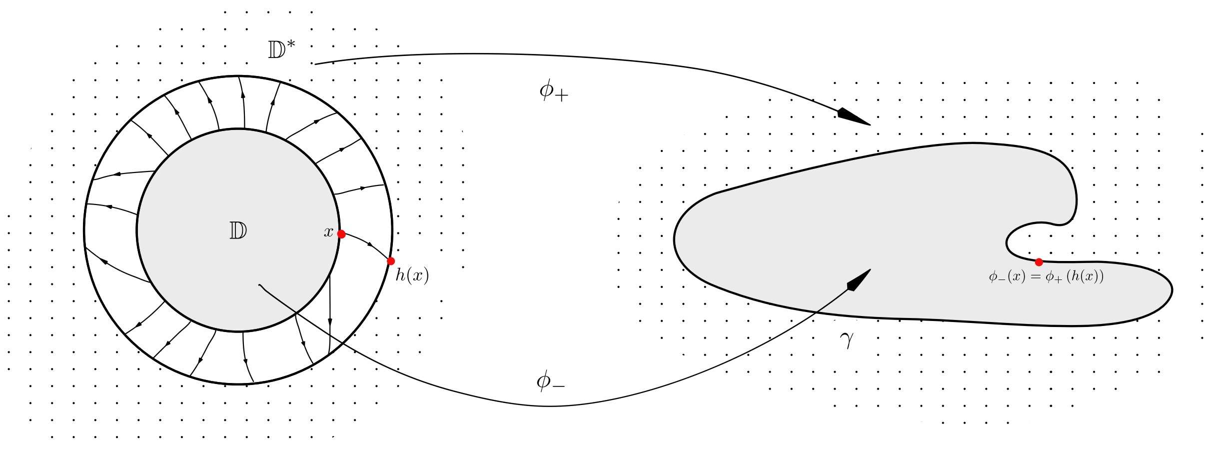


Figure 1.1: The left hand side depicts a homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}^*$. We have drawn $\partial\mathbb{D}^*$ slightly larger than $\partial\mathbb{D}$ so that the homeomorphism is visible. On the right hand side, a sketch of the curve γ, a solution to the welding problem, is shown. It has the property that $\phi_- = \phi_+ \circ h$ on $\partial\mathbb{D}$, where ϕ_- and ϕ_+ are Riemann maps onto the components of $\mathbb{C}\backslash\gamma$.

Conversely, let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be homeomorphism, which can be interpreted as an equivalence relation on the boundaries of D_1 and D_2. In this setting, the conformal welding problem is to find a Jordan curve $\gamma \subset \mathbb{C}$ for which h is a welding homeomorphism for γ. The resulting curve is said to be the[2] *conformal welding* of h. This can be related to the preceding discussion by using the solution ϕ_-, ϕ_+ to construct conformal charts for the topological quotient sphere $\overline{\mathbb{D}} \sqcup_h \overline{\mathbb{D}^*}$.

In this way we obtain a partial correspondence between homeomorphisms $h : S^1 \to S^1$ and Jordan curves in the plane (modulo some equivalence relations). For example, the homeomorphism $h(x) = x$ on S^1 corresponds to the curve $\gamma = \partial\mathbb{D}$.

For general homeomorphisms, it is difficult to determine whether a welding exists, and even when existence is known, it is not always clear that the solution is unique (up to Möbius

[2] Again, an abuse of language because the image of γ under any Möbius transformation of $\hat{\mathbb{C}}$ has the same welding homeomorphism h.

transformations).

1.1 Examples of Conformal Welding

In the examples below, instead of welding together $\mathbb{D}$ and $\mathbb{D}^*$ as above, we may also weld together the upper and lower half planes $\mathbb{H}_+, \mathbb{H}_-$. The definitions are similar, and the examples can be transported into the setting described above via Möbius maps. Here, the welding homeomorphism is a map $h : \mathbb{R} \to \mathbb{R}$ and a solution is an infinite arc γ in $\mathbb{C}$ with both endpoints at ∞.

Example 1.1.1. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^3$. Let $\gamma = [0, \infty) \cup i[0, \infty)$. Then $\phi_+(z) = z^{1/2}$ and $\phi_-(z) = z^{3/2}$ map $\mathbb{H}_+$ and $\mathbb{H}_-$ respectively onto the complementary components of $\mathbb{C}\backslash\gamma$. Furthermore, we have $\phi_+^{-1} \circ \phi_- = h$ on $\mathbb{R}$, so h is a welding homeomorphism for γ.

More generally, for $\alpha > 1$, $h(x) = x|x|^{\alpha-1}$ is a welding homeomorphism for $\gamma = [0, \infty) \cup e^{2\pi i(1+a)^{-1}}[0, \infty)$.

A small modification of the first example gives an example of a homeomorphism that has no welding solution. This example is even piecewise real analytic.

Example 1.1.2 ([49, Example 1])**.** Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x$ for $x < 0$, and $h(x) = x^3$ for $x \geq 0$. The welding problem for h does not have a solution. To see this, suppose for contradiction that $\gamma \subset \mathbb{C}$ is a solution and let ϕ_+ and ϕ_- be the conformal maps from the upper and lower half planes to the complementary components of $\mathbb{C}\backslash\gamma$ such that $h = \phi_+^{-1} \circ \phi_-$. We will use this to construct an analytic covering map $\pi : \mathbb{H} \to \mathbb{C}\backslash\{\gamma(0)\}$, which is a contradiction.

For integer k, define the conformal maps $\tau_k : \mathbb{R} \times (i3^k, i3^{k+1}) \to \mathbb{C}\backslash[0, \infty)$ by $\tau_k(z) = e^{2\pi \frac{z-3^k i}{3^{k+1}-3^k}}$ from the strip to the slit plane.

Next define $\pi_k(z) = \varphi_+ \circ \tau_k(z)$ if $z \in \mathbb{H}^+$ and $\pi_k(z) = \varphi_- \circ \tau_k(z)$ if $\tau_k(z) \in \mathbb{H}^-$, see Figure 1.2.

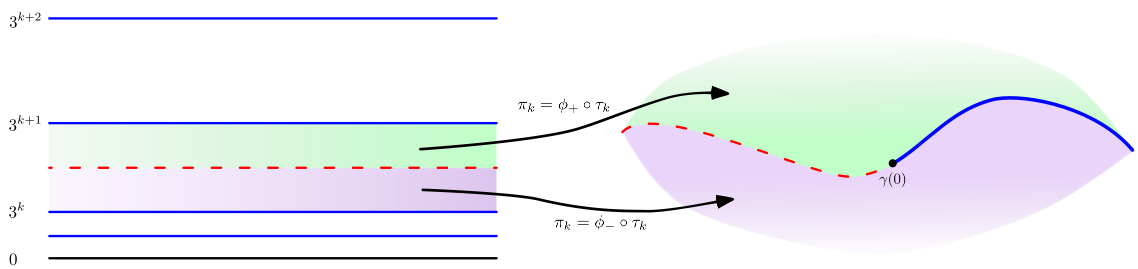


Figure 1.2: The conformal map π_k maps the strip $\mathbb{R} \times (i3^k, i3^{k+1})$ onto $\mathbb{C}\backslash\{\gamma(0)\}$, via the maps ϕ_+ and ϕ_-. The maps π_k and π_{k+1} agree on the line $\mathrm{Im}z = 3^{k+1}$, so the collection of maps $(\pi_k)_{k\in\mathbb{Z}}$ extends to a continuous map $\mathbb{H}^+ \to \mathbb{C}\backslash\{\gamma(0)\}$. Indeed this map is an analytic covering map, which is a contradiction.

Observe that the maps π_k and π_{k+1} extend to the boundary of their domains, and, because (ϕ_-, ϕ_+) is a solution the welding problem h, they are equal on the common boundary $\mathrm{Im}z = 3^{k+1}$. Thus the maps π_k glue together to form a continuous map π on the upper half plane $\mathbb{H} = \cup(\mathbb{R} \times [i3^k, i3^{k+1}])$. This map is a topological covering map $\pi : \mathbb{H} \to \mathbb{C}\backslash\{\gamma(0)\}$. By construction, the map is analytic except on a countable union of horizontal lines. By Morera's theorem, π is actually analytic on $\mathbb{H}$, and so π is an analytic covering $\mathbb{H} \to \mathbb{C}\backslash\{\gamma(0)\}$. This contradicts the fact that the universal covering space of $\mathbb{C}\backslash\{\gamma(0)\}$ is the plane and not the half plane.

Another way to derive a contradiction is to consider the modulus (see Section 2.3) of the family of loops surrounding $\gamma(0)$ and contained in a small neighbourhood of $\gamma(0)$. This modulus should be infinite, but one can show that if γ solves the welding problem h, then the modulus is actually finite, which is a contradiction. See [49, Example 1] for details.

If γ is a solution to the welding problem h, the question of whether this is the unique solution is closely related (but not necessarily equivalent, see [59, Question 1.2]) to the *conformal removability* of the set γ. We say that $K \subset \hat{\mathbb{C}}$ is *conformally removable for homeomorphisms* if the following implication holds: if $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism which is conformal on $\hat{\mathbb{C}}\backslash K$, then φ is conformal.

Proposition 1.1.3. *If γ is a solution to the welding problem h, and it is conformally removable for homeomorphisms, then in fact γ is the unique (up to Möbius) solution the welding problem.*

Proof. Let φ_- and φ_+ be the conformal maps into complementary components of $\hat{\mathbb{C}}\backslash\gamma$ such that $h = \varphi_+^{-1} \circ \varphi_-$. Suppose $\tilde{\gamma} \subset \hat{\mathbb{C}}$ is another solution, and let $\tilde{\varphi}_-$ and $\tilde{\varphi}_+$ be the corresponding conformal maps.

Define $\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\tau(z) = \begin{cases} \tilde{\varphi}_- \circ \varphi_-^{-1}(z) & \text{if } z \in \varphi_-(\overline{\mathbb{D}}) \\ \tilde{\varphi}_+ \circ \varphi_+^{-1}(z) & \text{if } z \in \varphi_+(\overline{\mathbb{D}^*}). \end{cases} \tag{1.1}$$

Then τ is well defined and continuous, and it is a homeomorphism, so by the definition of conformal removability, we have that τ is a Möbius transformation. □

Examples of welding homeomorphisms that do not have unique solutions can be found in [12].

We have seen that conformal welding gives a partial correspondence between homeomorphisms (up to Möbius equivalence) and Jordan curves in $\hat{\mathbb{C}}$ (up to Möbius equivalence). It is natural to ask for classes of homeomorphisms for which this correspondence is a bijection. The following classical result says that the welding problem for *quasisymmetric* homeomorphisms always has a unique solution. It also characterizes the geometry of the solution γ. A homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}^*$ is said to be K-quasisymmetric if $\mathrm{diam} h(I) \leq K \mathrm{diam} h(J)$ whenever $I, J \subset \partial\mathbb{D}$ are adjacent arcs of the same length. A Jordan curve γ is said to be a K-quasicircle if for all $x, y \in \gamma$, we have $\mathrm{diam}_{\mathbb{C}}(\gamma_{x,y}) \leq K|x - y|$. Here $\gamma_{x,y}$ is the smallest component of $\gamma\backslash\{x, y\}$.

All bi-Lipschitz homeomorphisms are quasisymmetric, and similarly all bi-Lipschitz loops are quasicircles. An example of a quasicircle is the Koch snowflake.

Theorem 1.1.4 (Fundamental Theorem of Conformal Welding [50])**.** *If h is quasisymmetric, then it has a unique (up to Möbius transformation) welding $\gamma \subset \mathbb{C}$. Moreover, a Jordan curve γ is a quasicircle if and only if its welding homeomorphism is quasisymmetric.*

In many modern applications, the quasisymmetry condition for conformal welding is not sufficient. In the calculus of random geometry developed by Duplantier, Miller and Sheffield, [22], the welding of 'random surfaces' (which are, roughly speaking, Riemann surfaces equipped with a certain random volume and boundary measures) along their boundaries is a fundamental operation. Here the welding homeomorphism $h : \partial S_1 \to \partial S_2$ is chosen so that the boundary measures ν_1, ν_2 of the surfaces are consistent, $h_* \nu_1 = \nu_2$.

Roughly speaking, the quasisymmetry condition is a regularity condition on each scale, however, for random homeomorphisms, it is almost always the case that the regularity condition will be violated at infinitely many scales. Recent work [7] in this direction has shown that 'regularity at most scales' suffices to show that the welding has a solution.

So far we have considered the problem of welding two disks (or half planes) along their boundaries via a homeomorphism to yield a sphere (or plane) with a distinguished simple curve. It is natural to generalize this and consider the welding problem for an arbitrary equivalence relation on the boundary of (the union of) Riemann surfaces.

For instance, if h is a homeomorphism from the upper semicircle $\partial\mathbb{D} \cap \overline{\mathbb{H}^+}$ to the lower semicircle $\partial\mathbb{D} \cap \overline{\mathbb{H}^-}$, this induces a welding problem where the solution is a simple arc γ which has the property $\phi = \phi \circ h$ on $\partial\mathbb{D}$, where $\phi : \mathbb{D}^* \to \mathbb{C}\backslash\gamma$ is a conformal map.

In this thesis we will be interested in the case when $\mathbb{D}^*$ is welded via an equivalence relation $\sim$ on $\partial\mathbb{D}^*$. A solution in this case is a compact connected set $K \subset \mathbb{C}$ for which the conformal map $\iota : \mathbb{D}^* \to \mathbb{C}\backslash K$ extends continuously to the boundary and $\iota(x) = \iota(y)$ iff $x \sim y$, see Figures 1.3, 1.7b and 1.8b. We will remove the Möbius freedom by requiring the normalization $\iota(z) = z + o(1)$ as $z \to \infty$.

The topology of the solution set K can be much more complicated than in the classical case of a Jordan curve, and indeed the possibility of 'branch points' presents additional

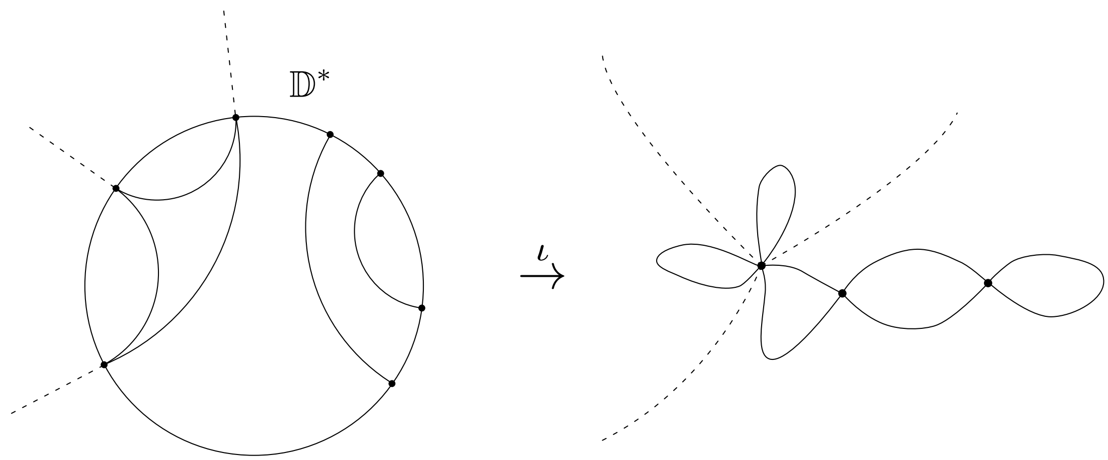


Figure 1.3: The left hand side depicts a equivalence relation $\sim$ on $\partial\mathbb{D}^*$. Points on $\partial\mathbb{D}^*$ are identified under $\sim$ if there is a curve in $\mathbb{D}$ connecting them. There are only finitely many (three) non-trivial equivalence classes. The conformal map ι on $\mathbb{D}^*$ extends to the boundary and makes the identifications given specified by $\sim$.

difficulties.

Example 1.1.5. If $\sim$ is an equivalence relation on $\partial\mathbb{D}^*$ such that each equivalence class is finite, and there are only finitely many nontrivial equivalence classes (Figure 1.3), then it not too hard to show that a welding exists, by using the classical Riemann uniformization theorem.

Note that in this case, the solution is highly non-unique (even with the normalization $\iota(z) = z + o(1)$). Given a solution we can construct another solution by the following procedure. Let $D \subset K$ be an open disk, and let φ be any non-identity conformal mapping on $\mathbb{C}\backslash K$ such that $\varphi(z) = z + o(1)$. Then $\varphi \circ \iota$ is another solution to the welding problem.

Example 1.1.6. Suppose $K \subset \mathbb{C}$ is a continuum, i.e. a locally connected, connected compact set with more than 2 points. Let $\iota : \mathbb{D}^* \to \mathbb{C}\backslash K$ be the unique conformal map with $\iota(z) = \lambda z + o(1)$ as $z \to \infty$, for some $\lambda > 0$. Suppose also that K has empty interior so that if $\iota : \mathbb{D}^* \to \mathbb{C}\backslash K$ is a conformal map fixing ∞, then $\iota(\partial\mathbb{D}^*) = K$. The *lamination* associated to K is the equivalence relation on $\partial\mathbb{D}^*$ defined by $x \sim y \iff \iota(x) = \iota(y)$. If K is conformally removable, then by the same argument as Proposition 1.1.3, $z \mapsto \iota(z/\lambda)$ is the unique (normalized) conformal map which solves the welding problem $\sim$.

Example 1.1.7. Let $p_c(z) = z^2 + c$ be a quadratic polynomial. Its *Julia set* is the set of points in $\mathbb{C}$ which are bounded under iteration by p_c:

$$J = \{z \in C : p_c^{\circ n}(z) \text{ is bounded in } n\}.$$

We say that p_c is *strictly preperiodic* if the orbit of the critical value c is pre-periodic but not periodic. It can be shown that the Julia set of such polynomials are dendrites. Thus this family of Julia sets provides many examples of weldable laminations, see Figure 1.7. In fact, the lamination can be computed easily without computing the conformal map. In Chapter 3 of this thesis, we investigate the regularity properties of this lamination.

Example 1.1.8 (True trees and Shabat polynomials)**.** Let T be a finite combinatorial plane tree with n edges. There is a natural way to represent T as a lamination (up to rotation), by dividing $\partial\mathbb{D}^*$ into $2n$ arcs of the same length and identifying pairs of arcs by an orientation reversing, arc length preserving homeomorphism, see Figure 1.5. We can use the uniformization theorem to prove that the welding exists, see Figure 1.6. Let $P_{2n} \subset \mathbb{C}$ be a regular polygon with $2n$ sides, viewed as a Riemann surface. This gives a model of the exterior unit disk $\mathbb{D}^*$, and there is a unique (up to rotation) conformal map $f : \mathbb{D}^* \to P_{2n}$ which maps ∞ to the center of P_{2n} and maps each of the $2n$ subarcs of $\partial\mathbb{D}^*$ onto the edges of P_{2n}.

Using this correspondence between edges of P_{2n} and subarcs of $\partial\mathbb{D}^*$, the lamination $\sim$ induces a pairing $\approx$ of the $2n$ edges of P_{2n}. The resulting quotient space is a Riemann surface homeomorphic to a sphere (the Riemann surface structure has only been defined the interior of the polygon, but it can be extended to the whole sphere by using flat charts on the edges of the polygon. See [28, Section 2.2] for more details). By the uniformization theorem there exists a conformal homeomorphism $\Phi : P_{2n}/\approx \to \hat{\mathbb{C}}$.

The composition $\iota := \Phi \circ f$ is the desired solution to the welding problem. Indeed, ι is unique up to Möbius transformations because $\iota(\partial\mathbb{D}^*)$ is a finite union of analytic arcs, which can be shown to be removable via Morera's theorem. Another proof for the existence of the solution to the welding problem, based on quasiconformal mappings, can be found in [14, Corollary 2.6].

If we normalize so that $\iota(z) = z + o(1)$, this procedure gives a unique *conformal embedding* of the combinatorial tree T.

Figure 1.4 shows the conformal embedding of a 3-regular tree of depth 12. Figure 1.8b shows a uniform random arc pairing lamination of $n = 50000$ edges, and an approximation of the solution to the corresponding welding problem. In Chapter 4, we prove that the solutions to the welding problem for a uniformly random arc pairing lamination with $2n$ edges converges (as $n \to \infty$) to a limiting object (Theorems 1.3.1 and 1.3.2), which we interpret as the conformal realization of the continuum random tree.

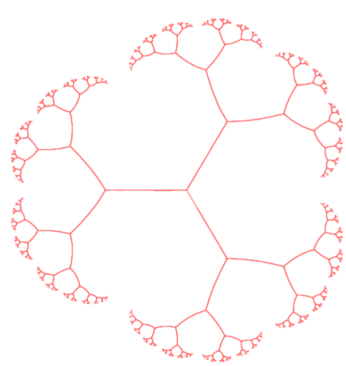


Figure 1.4: The conformal realization of the binary tree with 6141 edges. Generated with Donald Marshall's zipper software [45].

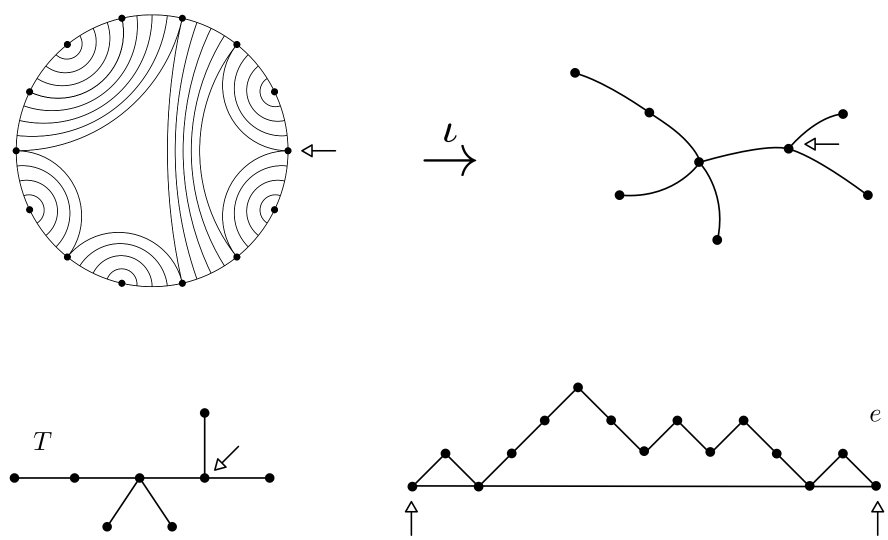


Figure 1.5: Bottom left: A combinatorial plane tree T with 7 edges. Bottom right: The representation of T as a Bernoulli excursion $e : [0, 1] \to \mathbb{R}_+$. Top left: The representation of T as a lamination $\sim$ on $\partial\mathbb{D}^*$. The unit circle is divided into 14 equally size arcs and pairs of arcs are identified with each other via orientation reversing and arclength preserving homeomorphism. Top right: A sketch of the solution K to the welding problem induced by $\sim$. The conformal map $\iota : \mathbb{D}^* \to \mathbb{C}\backslash K$ makes the identifications given by $\sim$, that is $x \sim y \iff \iota(x) = \iota(y)$.

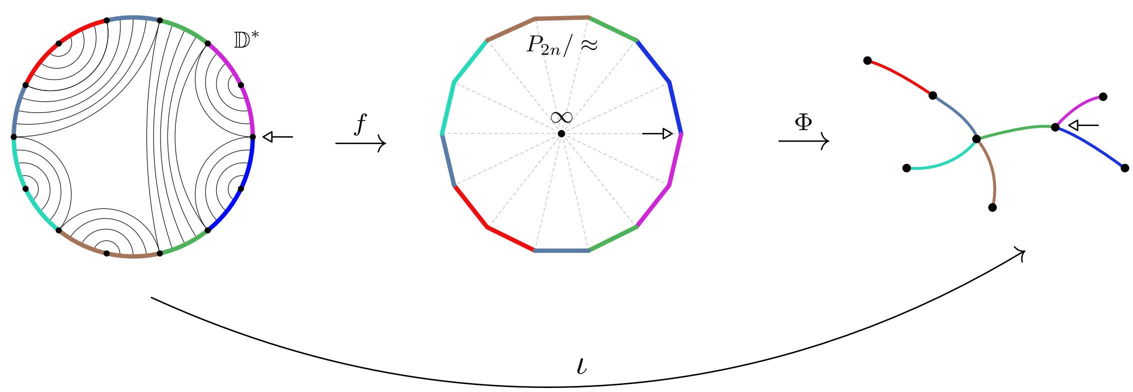
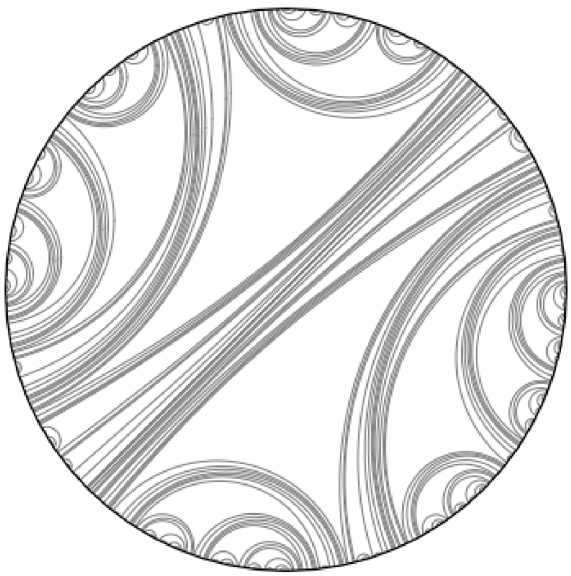
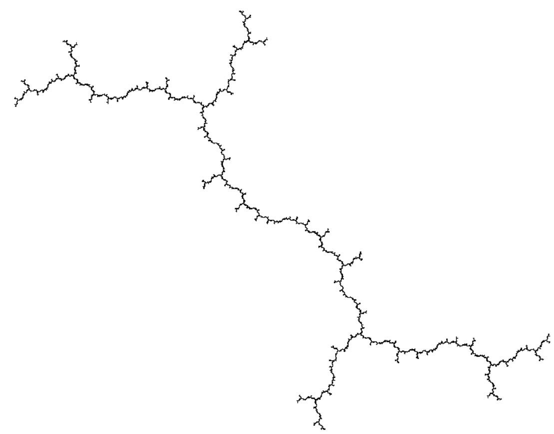


Figure 1.6: An arc pairing lamination $\sim$ of 14 arcs is depicted on the left. Such laminations can always be welded. Let P_{2n} be a regular polygon with $2n$ sides (here $n = 7$), viewed as a Riemann surface. Quotienting out by the relation $\approx$ induced by $\sim$ yields a Riemann surface $P_{2n}/\approx$ homeomorphic to a sphere, and the uniformization theorem gives a uniformizing map $\Phi : P_{2n}/\approx \to \hat{\mathbb{C}}$. There is a conformal map $f : \mathbb{D}^* \to P_{2n}/\approx$ which maps each of the 14 arcs to one of the 14 edges of P_{2n}. The composition $\iota = \Phi \circ f$ is the desired solution to the welding problem.



(a) The lamination $\sim_\alpha$ associated to $\alpha = 1/4$. The normalized Riemann map from the outside of the disk to the outside of the Julia set on the right identifies the endpoints of chords in this picture.



(b) The Julia set associated to the polynomial $p_c(z) = z^2 + c$ where $c \approx -.228 + 1.115i$ is a solution to the algebraic equation $p_c^2(c) = p_c^3(c)$.

Figure 1.7

1.2 Dendrite Julia Sets

We saw in Example 1.1.7 that Julia sets of certain quadratic polynomials p_c provide examples of weldable equivalence relations $\sim_c$ with branching. The relations $\sim_c$ are intricate and diverse for varying values of c, but there is a simple 'combinatorial' way of describing them without reference to any complex analysis, see [8, 37, 57] and Chapter 3. For every *combinatorial parameter* $\alpha \in \mathbb{T}/\{0\}$ there is an *abstract Julia equivalence* $\approx_\alpha$, and there is a way to relate c and α in such a way that, for many cases, $\sim_c = \approx_\alpha$.

There are several natural questions to ask about the welding relation $\approx_\alpha$.

Question 1.

- What is the relation between the geometric properties of the set J_c and the properties of the welding relation $\approx_\alpha$?

- Can it be seen directly that $\approx_\alpha$ has a welding solution? Of course we already know that $\sim_c$ has a welding solution, but we are asking if it is possible to see this directly from the combinatorial construction of $\approx_\alpha$.

One of the main theorems in [44] is a tree-welding analog of the 'Fundamental Theorem of Conformal Welding (for curves)' (Theorem 1.1.4). It is proved that a welding relation has a *Gehring tree* solution if and only if the welding has 'quasisymmetric gluings at all scales and all locations'. A Gehring tree is a dendrite $K \subset \mathbb{C}$ such that the complement $\mathbb{C} \backslash K$ is a *John domain*; this means that every point in the domain can be joined to any other point in the domain without going too close to the boundary. A quasisymmetric gluing is, roughly speaking, a chain of pairs of intervals around x for which a thick (uniformly perfect) subset of each interval is identified quasisymmetrically with a thick subset of the other interval in its pair. See Theorem 2.5.3 for details.

In [17] it is shown that Fatou sets for *semihyperbolic* polynomials are John domains. The semihyperbolicity condition can be defined as a dynamical condition on the orbit of the critical points under iteration of the polynomial.

Theorem 1.2.1 ([17]). *The polynomial p is semihyperbolic if and only if the complement of the filled Julia set is a John domain.*

In Chapter 3, we formulate the semihyperbolicity condition purely in terms of the combinatorial parameter $\alpha \in \mathbb{T}$ (and call it *combinatorial semihyperbolicity*), and we prove that the relations $\approx_\alpha$ corresponding to combinatorially semihyperbolic parameters satisfy the quasisymmetric gluing condition described above.

Theorem 1.2.2. *[43, Theorem 1.2] Suppose α is combinatorially semihyperbolic. Then $\approx_\alpha$ satisfies the hypotheses of Theorem 2.5.3.*

This result illustrates how the quasisymmetric gluing criterion of [44] can be useful for solving the welding problem. It inspired our further work on random welding problems, described in Section 1.3.

We also show that our notion of combinatorial semihyperbolicity matches the notion of semihyperbolicity [17] in the concrete setting.

Theorem 1.2.3. *[43, Theorem 5.1] $\alpha \in \mathbb{T}$ is combinatorially semihyperbolic if and only if the associated $c \in \mathbb{C}$ is semihyperbolic and J_c is a dendrite Julia set.*

Our proof is purely combinatorial, so together with the results of [44], which say that quasisymmetric weldings have unique solutions and the solutions are *Gehring trees* - which are a generalization of quasicircles to the dendritic setting - we get an independent proof of Theorem 3.1.4 in the case of dendritic quadratic polynomials.

Corollary 1.2.4. *If α is combinatorially semihyperbolic, the welding problem for $\approx_\alpha$ has a unique solution and the complement of the solution is a Gehring tree.*

Theorem 1.2.2 describes the connection between John domains and semihyperbolic dynamics, and we proved the correspondence 'semihyperbolic equivalence' $\leftrightarrow$ 'quasisymmetric lamination'.

In future work, we would like to extend the results of this thesis to cover the Collet-Eckmann (CE) [52, 55, 29] quadratic polynomials, which can have non-quasisymmetric geometry.

1.3 Conformal embedding of random trees

The Brownian lamination is a random equivalence relation on $\mathbb{T}$ obtained from the Brownian excursion as follows. Let $e : [0,1] \to [0,\infty)$ be any continuous function with $e(0) = e(1) = 0$, i.e. an *excursion*. This induces a pseudometric $d_e(\cdot,\cdot)$ on $[0,1]$ by

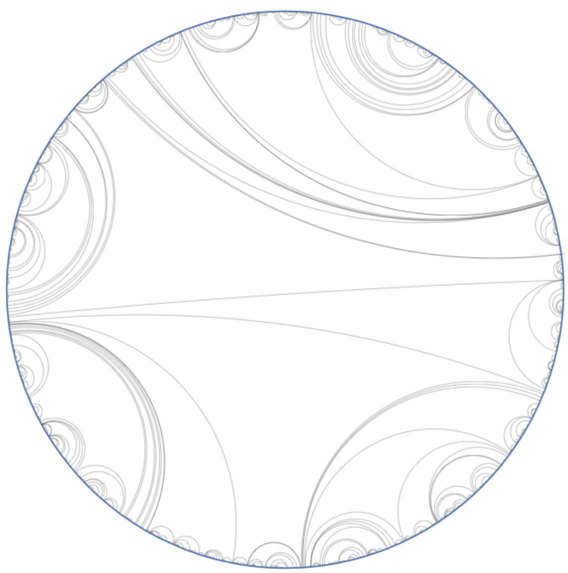
$$d_e(x,y) = e(x) + e(y) - 2 \min_{t\in[x,y]} e(t), \qquad x, y \in [0,1]. \tag{1.2}$$

Define $x \sim_e y$ if and only if $d_e(x,y) = 0$. In particular $0 \sim_e 1$, so we can view $\sim_e$ as an equivalence relation on the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. If e is taken to be the standard Brownian excursion, then the resulting random equivalence relation $\sim_e$ is called the Brownian lamination. The quotient of $\mathbb{T}$ under the Brownian lamination is known as the continuum random tree (CRT) [3].

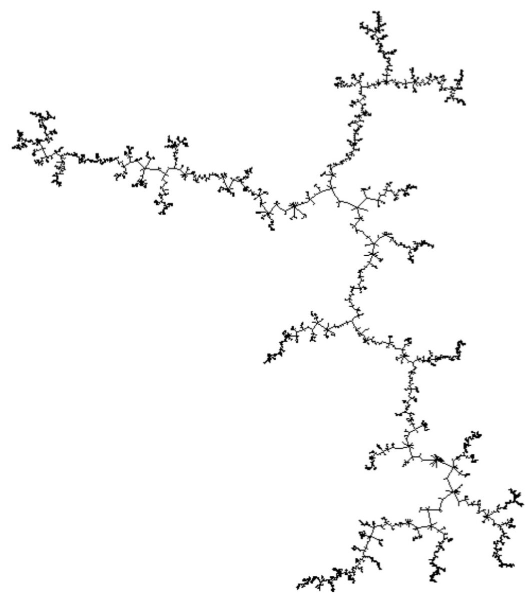
It is easy to show that the CRT does not satisfy the quasisymmetry hypothesis of Theorem 2.5.3. However, we prove that the Brownian lamination can still be welded.

Theorem 1.3.1. *Almost surely, the Brownian lamination admits conformal welding. The solution is unique up to Möbius transformation, and it is almost surely Hölder continuous with a deterministic universal exponent* $\alpha_0 > 0$.

Rather than proving this theorem directly, we prove a stronger result regarding convergence of the Shabat trees introduced in Example 1.1.8. Shabat trees are the special case of Grothendieck's dessin d'enfant where the graph is a tree drawn on the sphere so that the Belyi function is a polynomial, known as a Shabat polynomial or generalized Chebychev polynomial. They arise in a number of different ways, and in particular can be shown ([11, 14]) to be the dendrites whose laminations are given by non-crossing pairings of $2n$ arcs of equal size on the circle (see Example 1.1.8). Equivalently, these laminations are given by the Dyck



(a) A sample of $\sim_{e_n}$ for $n = 30000$. This is generated by dividing the unit circle into $2n$ arcs of equal size and choosing a uniformly random non-crossing pairing of the arcs. As $n \to \infty$, this equivalence relation converges to the Brownian lamination.



(b) The solution to the welding problem $\sim_{e_n}$. The Riemann map from the outside of the disk to the outside of this set identifies the endpoints of chords in the left picture. We prove that as $n \to \infty$, this converges to the welding solution to the Brownian lamination. Generated by Donald Marshall's zipper software [45].

Figure 1.8

paths S coding the trees via (1.2), see Figure 1.5. Thus a Shabat tree with n edges chosen uniformly at random can be viewed as the welding solution to the lamination associated with simple random walk excursions of length $2n$ via (1.2). Roughly speaking, we prove that the (suitably normalized) uniform random Shabat tree of n edges converges (distributionally in the Hausdorff topology, and even stronger the topology induced by conformal parametrization) to the random dendrite of Theorem 1.3.1. More precisely, it is not hard to show that the laminations induced by simple random walk excursions converge to the Brownian lamination in distribution (similar to the convergence of rescaled simple random walk to Brownian motion), and we show that the solutions to the welding problems also converge.

Theorem 1.3.2. *There exists a deterministic universal constant $\alpha_0 > 0$ such that the following holds. If $\iota_n : \mathbb{D} \to \mathbb{C}$ is the welding map for the uniform random lamination (the random non-crossing pairing of $2n$ arcs), then ι_n converges in distribution to a (random) conformal map ι, with respect to uniform convergence on $\overline{\mathbb{D}}$. Furthermore, ι is almost surely α_0-Hölder continuous. The law of the associated lamination $\widetilde{L}_\iota$ is that of the Brownian lamination.*

1.4 Related Work

1.4.1 Conformal Welding of Jordan curves and trees

When two disks are welded together via a homeomorphism, existence and uniqueness of the conformal welding is guaranteed if h is quasisymmetric [41, 50], and this is sometimes referred to as the 'fundamental theorem of conformal welding' [31]. However, a full characterization of welding homeomorphisms seems difficult - for instance, see the results and references in [13]. In [7], it was shown that certain random homeomorphisms can be almost surely welded, yielding a probability measure on the space of loops modulo Möbius transformations. Note that the random homeomorphisms considered there are almost surely not quasisymmetric, so that the fundamental theorem of conformal welding cannot be used to deduce existence and uniqueness of the solution.

In this thesis, we investigate the welding of the boundary of a single disk, where the welding data is specified by a lamination (equivalence relation) on S^1, and the resulting surface is homeomorphic to the sphere. The theses [42] and [30] give sufficient conditions for the existence of a solution - namely that solutions exist when only a small (zero capacity) set of points have nontrivial equivalence class.

The thesis [9] also investigates the Brownian lamination. It is proved that any subsequential limit ι of the finite solutions ι_n must be nontrivial. That is, $\mathbb{P}(\iota = \mathrm{Id}) = 0$. Bishop [14] proves the deterministic result that any compact set in the plane can be approximated by the compact set $\iota(\partial\mathbb{D})$, where ι is the welding for some arc pairing lamination $\sim_{e_n}$.

1.4.2 Conformal representation of (large) triangulations

Let $\mathcal{T}$ be a finite triangulation of the sphere S^2. We may construct a Riemann surface from the data of $\mathcal{T}$ by gluing equilateral triangles along edges according the combinatorics of $\mathcal{T}$. One way to make this precise is to interpret this as the conformal welding problem where $S = \bigsqcup_{i=1}^n \Delta_i$, and $\sim$ identifies edges of different Δ_i. In this case the uniformization theorem implies that the welding exists. That is, we can define the Riemann surface structure on $\overline{S}/\sim$ by writing down charts at each point explicitly, and it is not hard to see that the structure is unique. See [28] for details. The uniformization theorem implies the existence of a homeomorphism $\iota : \overline{S}/\sim \to \hat{\mathbb{C}}$ which is conformal on the union of the triangles S. This gives a canonical way of drawing triangulations of the sphere (up to an Möbius transformation of $\hat{\mathbb{C}}$). Also, the pushforward of the natural Euclidean metric and Lebesgue measure on S via ι induces a corresponding metric and measure on $\hat{\mathbb{C}}$.

This procedure provides a way to map a (combinatorial) triangulation of S^2 to a metric/measure space structure on $\hat{\mathbb{C}}$ (modulo Möbius transformations). This procedure also provides a canonical rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with critical values in $\{0, 1\infty\}$, where the preimage $f^{-1}(\mathbb{R})$ is exactly the triangulation $\mathcal{T}$. Such rational maps are called Bélyi functions.

This correspondence between triangulations and Bélyi functions is of interest in algebraic geometry and number theory [53, 39]. In recent years, the properties of the corresponding metric/measure/Riemann surface structure for large random triangulations have been studied [18, 10, 28]. One motivation for this interest is the conjecture [54, 19] that measures and metrics associated to large random triangulations will converge to certain continuum models as the size of the triangulations goes to ∞. See [26] for an exposition of this area.

From the construction in Example 1.1.8, we see that welding problem for uniform random laminations $\sim_{e_n}$ can be viewed as a certain random triangulation (the polygon P_{2n} used there can be replaced with the gluing of $2n$ equilateral triangles identified around a common vertex).

1.4.3 Continuum random tree and mating

The random metric space $(S^1/\sim, d_e)$, introduced by [3, 4, 5] is known as the *continuum random tree* (CRT). Our definition corresponds to Corollary 22 in [5]. Being the scaling limit of many models of random finite trees, it is important and well studied in probability theory, see the survey [40]. In [22], it is shown that two independent CRTs can be glued together to form a topological sphere equipped with a canonical measure and space filling curve, and that this quotient space can be canonically embedded in S^2 in such a way that the measure is Liouville Quantum Gravity and the curve is space filling SLE. It would be interesting to see whether we can use our conformal embedding of the CRT (Theorem 1.3.1) to provide another construction of this embedding via an analogy of the mating construction [15] of complex dynamics.

Chapter 2
PRELIMINARIES

In this chapter we collect definitions and facts about quasisymmetric maps, logarithmic capacity, conformal modulus and uniformly perfect sets, as can be found for instance in the monographs [1], [2], [6], [27], [32], [51].

2.1 Notation

Throughout this thesis we will use the following notation: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, $\mathbb{D}$ is the (open) unit disc, $\mathbb{D}^* = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle.
We write $a \lesssim_\lambda b$ to designate the existence of a function $C(\lambda)$ such that $a/b \leq C(\lambda)$, and $a \lesssim b$ if $a \leq Cb$ for some constant $C > 0$. We write $a \asymp_\lambda b$ to mean that $a \lesssim_\lambda b$ and $b \lesssim_\lambda a$.

2.2 John domains, quasisymmetric maps and Gehring trees

A connected open subset D of the Riemann sphere is a *John-domain* if there is a point $z_0 \in D$ (the John-center) and a constant C (the John-constant) such that for every $z \in D$ there is a curve $\gamma \subset D$ from z_0 to z such that

$$\mathrm{dist}(\gamma(t), z) \leq C\mathrm{dist}(\gamma(t), \partial D)$$

for all t. If $\infty \in D$, then $z_0 = \infty$. An equivalent definition ([51]) is that

$$\mathrm{diam}D(\sigma) \leq C'\mathrm{diam}\sigma \tag{2.1}$$

for every crosscut σ of D, where $D(\sigma)$ denotes the component of $D \setminus \sigma$ that does not contain z_0. Moreover, it is enough to consider crosscuts that are line segments.

John domains were introduced in [34] and are ubiquitous in analysis. Simply connected planar John domains can be viewed as one-sided quasidiscs. Indeed, a Jordan curve is a quasicircle if and only if both complementary components are John domains. Important work related to John domains can be found in [6],[35],[17],[48],[58] and a large number of references in these works. A *(planar) dendrite* is a compact, connected, locally connected subset T of the plane $\mathbb{C}$ with trivial fundamental group.

Definition 2.2.1. A Gehring tree is a planar dendrite such that the complement is a John-domain.

The notion of *quasisymmetry* is a generalization of quasiconformality to the setting of metric spaces, see [32]. An embedding $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $d_Y(\phi(x), \phi(z)) \leq \eta(t)d_Y(\phi(y), \phi(z))$ whenever $d_X(x, z) \leq td_X(y, z)$. In this thesis we will be mostly be concerned with the case when f is a homeomorphism and X, Y are subsets of the interval or the circle. Here, a homeomorphism $f : X \to Y$ is quasisymmetric iff there exists $K > 1$ such that $|\phi(x) - \phi(y)| \leq K|\phi(y) - \phi(z)|$ whenever $|x - y| \leq |y - z|$.

2.3 Logarithmic capacity, conformal modulus and uniformly perfect sets

Let μ be a Borel measure of finite nonzero mass on $\mathbb{C}$. The *(logarithmic) energy* $\mathcal{E}(\mu)$ of the measure μ is the extended real number

$$\mathcal{E}(\mu) = \iint_{\mathbb{C}^2} - \log |x - y| \, d\mu(x)d\mu(y). \tag{2.2}$$

The *(logarithmic) capacity* of a compact set $E \subset \mathbb{C}$ is the real number

$$\mathrm{cap}(E) = e^{-\inf_\mu \mathcal{E}(\mu)},$$

where the infimum is taken over all Borel probability measures supported on E. Important examples are $\mathrm{cap}\, B(x, r) = r$ and $\mathrm{cap}[a, b] = |b - a|/4$. The capacity of a Borel set is defined as the supremum of the capacities of compact subsets.

Now let $\Omega \subset \mathbb{C}$ be an open set. A *path* in Ω is a countable collection of rectifiable curves in Ω. A *path family* in Ω is a collection of paths in Ω.

A *(conformal) metric* on Ω is a measurable nonnegative extended real valued function $\rho : \Omega \to [0, \infty]$. The *area* of a metric is the quantity $\text{Area}(\rho) = \iint_\Omega \rho(x, y)^2 dxdy$.

If γ is a path in Ω then the ρ-length of γ is $\ell_\rho(\gamma) = \int_\gamma \rho\, ds$.

The *modulus* $\text{Mod}(\Gamma)$ of a path family Γ is the quantity

$$\text{Mod}(\Gamma) = \inf_\rho \text{Area}(\rho) \tag{2.3}$$

where the infinimum is taken over all *admissible* metrics ρ, namely Borel measurable functions ρ such that $\inf_{\gamma\in\Gamma} \ell_\rho(\gamma) \geq 1$.

Example 2.3.1. Let $R = [0, 1] \times [0, M]$ be the rectangle with aspect ratio $M > 0$. Let Γ be the family of paths joining the top edge of R to the bottom edge of R. Then by setting $\rho \equiv 1/M$ in the definition (2.3), we see that $\text{Mod}(\Gamma) \leq 1/M$. In fact, we can show that $\text{Mod}(\Gamma) \geq 1/M$ too. Let $\rho : R \to [0, \infty]$ be an admissible conformal metric, then for all $x > 0$ we have

$$\int_0^M \rho(x + iy)dy \geq 1,$$

and the Cauchy-Schwarz inequality implies

$$M \int_0^M \rho(x + iy)^2 dy \geq 1.$$

Integrating this over $x \in [0, 1]$ gives

$$M \int_R \rho(x + iy)^2 dxdy \geq 1,$$

i.e. $\text{Area}(\rho) \geq 1/M$. Since ρ was arbitrary, this proves that $\text{Mod}(\Gamma) \geq 1/M$ as desired. Similarly, if Γ^* denotes the family of paths from the left edge of R to the right edge of R, one can show that $\text{Mod}(\Gamma^*) = M$.

If $f : \Omega \to \Omega'$ is a conformal map, and $\rho : \Omega' \to \mathbb{R}$ is a metric on Ω', define the pullback metric $f^*\rho : \Omega \to \mathbb{R}$ to be the metric $f^*\rho(z) = \rho(f(z))|f'(z)|$. Then $\text{Area}(f^*\rho) = \text{Area}(\rho)$, and $\ell_{f^*\rho}(\gamma) = \ell_\rho(f \circ \gamma)$ for any path γ in Ω.

Let $A \subset \mathbb{C}$ be a topological annulus, that is, A is connected, open and has exactly two complementary components. The *modulus* of A is defined to by $\mathrm{Mod}(A) = \mathrm{Mod}(\Gamma)$ where Γ is the family of simple loops in A that separate the components of $\mathbb{C}\backslash A$. By the discussion above, $\mathrm{Mod}(A) = \mathrm{Mod}(f(A))$ if f is conformal on A. If $A = \{z : r < |z| < R\}$ is a round annulus for some $0 < r < R < \infty$, then $\mathrm{Mod}(A) = \frac{1}{2\pi}\log(R/r)$ (the proof is similar to that of Theorem 2.3.1).

The following lemma relates diameter and modulus. Together with Lemma 2.3.3, this gives a useful method for getting upper bounds on the diameters of sets.

Lemma 2.3.2. *Suppose $E \subset \mathbb{C}$ is compact, $D \subset \mathbb{C}$ is open, and $E \subset D$ so that $D\backslash E$ is a non-degenerate annulus. Then*

$$\mathrm{diam} E < 2 \cdot 2^{-\mathrm{Mod}(D\backslash E)} \mathrm{diam}(D). \tag{2.4}$$

Proof. We get can get a sharper result by using the asymptotics for the solution to Teichmüller's extremal problem [1, Chapter 3], but we will present a simpler argument that uses essentially just the definition of modulus. This proof is adapted from [46]. Consider the constant metric $\rho \equiv 1$ on $D\backslash E$ and fix $\epsilon > 0$. By the definition of modulus, there exists a simple loop $\gamma \subset D\backslash E$ separating E from $\mathbb{C}\backslash D$ such that

$$\frac{\ell_\rho(\gamma)^2}{\mathrm{Area}(\rho)} \le \mathrm{Mod}(D\backslash E)^{-1} + \epsilon. \tag{2.5}$$

By the isodiametric inequality [24, Theorem 2.4], we have $\mathrm{Area}(\rho) = \iint_{D\backslash E} dxdy \le \iint_D dxdy \le \frac{\pi}{4}(\mathrm{diam} D)^2$. We also have $\ell_\rho(\gamma) \ge 2\mathrm{diam} E$, so from (2.5) and taking $\epsilon \to 0$ we get

$$4(\mathrm{diam} E)^2 \le \frac{\pi}{4}(\mathrm{diam} D)^2 \mathrm{Mod}(D\backslash E)^{-1}.$$

Thus

$$\mathrm{diam} E \le \frac{\sqrt{\pi}}{4}\mathrm{diam} D \cdot \mathrm{Mod}(D\backslash E)^{-1/2} < \frac{1}{2}\mathrm{diam} D \cdot \mathrm{Mod}(D\backslash E)^{-1/2}. \tag{2.6}$$

This proves the desired inequality when $\mathrm{Mod}(D\backslash E) = 1$.

For the general case, divide the annulus $D\backslash E$ into $\lceil \text{mod}(D\backslash E)^{-1} \rceil$ topologically concentric topological annuli $D\backslash E_1$, $E_1^\circ\backslash E_2$, $E_2^\circ\backslash E_3, \ldots, (E^\circ_{\lceil \text{mod}(D\backslash E)^{-1}\rceil} - 1)\backslash E$ such that the first $\lfloor \text{mod}(D\backslash E)^{-1} \rfloor$ annuli have modulus equal to 1. This can be done by first mapping $D\backslash E$ conformally onto a round annulus, dividing this round annulus into concentric round annuli of the correct modulus, and then mapping back to $D\backslash E$.

Iterating (2.6) yields $\text{diam}E \leq 2^{-\lfloor \text{mod}(D\backslash E)^{-1}\rfloor}\text{diam}D$ and this implies the desired inequality.

□

The following simple lemma allows us to bound the modulus of a large annulus by bounding the modulus of a collection of smaller annuli contained inside.

Lemma 2.3.3. *Let A be a topological annulus, and let $A_1, \ldots, A_N$ be a sequence of annuli such that*

- *For each $1 \leq i \leq N-1$, the annulus A_{i+1} is contained in the bounded component of $\mathbb{C}\backslash A_i$*
- *Each A_i is* essentially embedded *in A: we have $A_i \subset A$, and if $\gamma \subset A_i$ is a loop separating the boundary components of A_i, then γ separates the boundary components of A.*

Then

$$\text{Mod}(A) \geq \sum_{i=1}^{N} \text{Mod}(A_i).$$

Proof. Let Γ_i be the loops in A_i that separate the boundary components of A_i, and let Γ be the loops in A that separate the boundary components of A. Fix $\epsilon > 0$ and let ρ be a metric on A such that $\inf_{\gamma\in\Gamma} \int_\gamma \rho ds \geq 1$ and $\text{Area}(\rho) \leq \text{Mod}(\Gamma) + \epsilon$. Let $\rho^i = \rho|_{A_i}$. Then $\inf_{\gamma\in\Gamma_i} \int_\gamma \rho^i ds \geq 1$ because $\Gamma_i \subset \Gamma$, so $\text{Mod}(\Gamma_i) \leq \text{Area}(\rho^i)$ for all i. Thus

$$\sum \text{Mod}(A_i) \leq \sum \text{Area}(\rho^i) \leq \text{Area}(\rho) \leq \text{Mod}(\Gamma) + \epsilon.$$

The middle inequality is due to the fact that all the Γ_i are disjoint. Taking $\epsilon \to 0$ yields the result. □

The previous two lemmas are frequently combined to show that a given compact set $K \subset \mathbb{C}$ is small. We will be using this lemma with $K = \iota(I)$ where ι is the welding map and $I \subset \mathbb{T}$ is an arc.

Lemma 2.3.4. *Suppose $K \subset \mathbb{C}$ is contained in the disk of radius R, $\{z : |z| < R\}$. Suppose there exists $\delta > 0$ and a sequence of topological annuli $A_1, A_2, \ldots, A_N$ such that*

- $\mathrm{Mod}(A_i) > \delta$.
- *The A_i are contained in the disk $\{z : |z| < R\}$.*
- *For $i = 1, \ldots, N-1$, the annulus A_{i+1} is contained in the bounded component of $\mathbb{C} \backslash A_i$.*
- *K is contained in the bounded component of $\mathbb{C} \backslash A_N$.*

Then $\mathrm{diam} K \leq 2 \cdot 2^{-N\delta} R$.

Proof. Combine Lemmas 2.3.2 and 2.3.3. □

We will also use Pfluger's theorem which quantifies a close connection between capacity and modulus, see [51, Theorem 9.17] for a proof:

Theorem 2.3.5. *If $B \subset \partial\mathbb{D}$ is a Borel set and if Γ_B is the set of all curves $\gamma \subset \mathbb{D}$ joining the circle C_r to the set B, then*

$$\operatorname{cap} B \asymp e^{-\pi/M(\Gamma_B)}$$

with the implicit constants depending only on $0 < r < 1$.

Specifically, we will use the following variant, which we leave as an exercise for the reader.

Theorem 2.3.6. *If $S = [0,1] \times [0, M]$ is a rectangle, if $B \subset [0,1] \times \{M\}$ is a Borel subset of the top edge, and if Γ_B is the family of curves that join the bottom edge $[0,1] \times \{0\}$ to B in S, then*

$$\operatorname{cap} B \leq C(M) e^{-\pi/2M(\Gamma_B)}. \tag{2.7}$$

A compact set A is called *uniformly perfect* if there is a constant $c > 0$ such that no annulus $A(x, cr, r)$ with $r < \mathrm{diam} A$ separates A: If $A \cap A(x, cr, r) = \emptyset$, then $A \subset B(x, cr)$ or $A \cap B(x, r) = \emptyset$. See [27, Exercise IX.3] for 13 other equivalent definitions.

2.4 Modulus of welded rectangles and annuli

In this section we develop one of the main analytic tools of this thesis. The main result of this section provides an estimate for the modulus of conformally welded rectangles in terms of the quality of the welding on a small subset of the welded boundary.

Let $I^+ = [0, 1]$ and $I^- = [0, 1] - i$. Let $S^+ \subset \mathbb{C}$ be the square with sides parallel to the axes, with bottom edge I^+. Let $S^- \subset \mathbb{C}$ be the square with top edge I^-. Now let $h : I^+ \to I^-$ be a homeomorphism and suppose that h is a welding homeomorphism, meaning that there exists a homeomorphism $\iota : (S^+ \sqcup S^-)/\sim_h \to \Omega$ where Ω is a simply connected domain and ι is conformal on S^+ and S^-. Without loss of generality (by composing with a Riemann map) we can assume that Ω is a rectangle $R = [0, 1] \times [0, M]$, and that the four vertices of $(S^+ \sqcup S^-)/\sim$ are mapped to the vertices of R in the following way: $i \mapsto Mi$, $i + 1 \mapsto 1 + Mi$, $-2i \mapsto 0$ and $1 - 2i \mapsto 1$. We are interested in bounds on the modulus M.

If h is the map $h(x) = x - i$ then $M = 2$. On the other hand, if h is less well-behaved, then it is possible for M to be arbitrarily large.

In our work it is crucial to find a criterion on h which ensures control over M. The classical quasiconformal theory gives one such criterion. (We will not prove either of the following propositions since we do not need them. They also follow from Proposition 2.4.4).

Proposition 2.4.1. *If $h : I^+ \to I^-$ is K-quasisymmetric, then M is bounded by some function of K.*

In our setting of dendrites, the existence of branch points means we need control even when I^+ is not glued to I^- via a homeomorphism, see Figure 2.1a. The following proposition gives a criterion that works in this setting. It shows that as long as sufficiently 'large' parts of I^- and I^+ are identified via a quasisymmetric homeomorphism, we can get control over M.

Proposition 2.4.2. *Let $\sim$ be an equivalence relation on $I^+ \cup I^-$. Let E^+ and E^- be subsets of I^+ and I^- respectively and let $h : I^+ \to I^-$ be a homeomorphism such that $x \sim h(x)$ for $x \in E^+$. If E^+ is uniformly perfect and h is quasisymmetric, then M is bounded above by a function of the quasisymmetry constant of h and the uniform perfectness constant of E^+.*

For this reason, we say that $\sim$ is *L-quasisymmetrically thick* between I^- and I^+ if the hypothesis of the preceding proposition holds (here L is a parameter quantifying the uniform perfectness constant and quasisymmetry constant).

However, this is still not strong enough for our purposes. As mentioned in the introduction, the notions of uniform perfectness and quasisymmetry are not sufficient in random settings since 'almost surely something bad will happen at some scale somewhere'. We need the following lemma, which allows us to deal with the random setting.

The proposition provides an estimate for the modulus of conformally welded rectangles in terms of the quality of the welding on a small subset of the welded boundary.

Definition 2.4.3. Let $\sim$ be an equivalence relation on $I^- \cup I^+$. A pair μ^-, μ^+ of probability measures supported on I^- and I^+ respectively is called a *gluing pair* for $\sim, I^-, I^+$ if there is a measure preserving bijection ϕ between measurable subsets $E^- \subset I^-$ and $E^+ \subset I^+$ of full measure, $\mu^-(E^-) = \mu^+(E^+) = 1$, such that $\phi(x) \sim x$ for all $x \in E^-$.

A *topological rectangle* is a Jordan domain where two disjoint arcs on the boundary have been marked. In the following statement, S^+ and S^- are squares, but it is clear that we can use Riemann maps to generalize to the case when S^+ and S^- are topological rectangles, in which case the constant C_0 of the conclusion depends on the geometry of S^+ and S^-. We will use this generalization freely in Section 2.5 to get bounds on the welding of topological rectangles.

Proposition 2.4.4. *The family Γ of pairs of paths (γ^+, γ^-) joining the top edge of S^+ to the bottom edge of S^- such that the endpoint of γ^+ is equivalent to the initial point of γ^- satisfies*

$$\mathrm{Mod}(\Gamma)^{-1} \le C_0 \inf_{\mu^-, \mu^+} \max\left(\mathcal{E}(\mu^-), \mathcal{E}(\mu^+)\right), \tag{2.8}$$

where the infimum is taken over all gluing pairs of measures as above, and $\mathcal{E}$ *denotes logarithmic energy.*

Note that the quantity $\mathrm{Mod}(\Gamma)^{-1}$ in the lemma above is an upper bound for the aspect ratio M of the welded squares.

Proof. Let $\rho = (\rho^-, \rho^+)$ be a conformal metric on the disjoint union $S^- \sqcup S^+$ and let μ^-, μ^+ be Borel probability measures supported on I^- and I^+ respectively, together with a measure preserving bijection ϕ as in Definition 2.4.3. Let $B^+ \subset E^+$ be the set of *bad* points p such that every curve $\gamma^+ \subset S^+$ from the top edge of S^+ to p has ρ^+-length at least $1/2$, and define $B^- \subset E^-$ similarly so that all the curves from E^- to the bottom edge of S^- have ρ^--length at least $1/2$. By definition we have that $\mathrm{Mod}(\Gamma_{B^i}) \leq 4\mathrm{Area}(\rho^i)$ for $i = -, +$. By Pfluger's Theorem 2.3.6 we have

$$\operatorname{cap} B^i \leq C_0 e^{-\pi/(2\mathrm{Mod}(\Gamma_{B^i}))} \leq C_0 e^{-\pi/(8\mathrm{Area}(\rho^i))} \leq e^{-\pi/(8\mathrm{Area}(\rho^i))+c_0}$$

where we assume without loss of generality that $C_0 \geq 1$ and $c_0 \geq 0$. Suppose that

$$\mathrm{Area}(\rho^i) \leq \frac{\pi/8}{16\ \mathcal{E}(\mu^i) + c_0} \text{ for both } i = -, + \tag{2.9}$$

so that $\operatorname{cap} B^i \leq \exp(-16\ \mathcal{E}(\mu^i))$. Then

$$\mu^i(B^i) \leq 1/4. \tag{2.10}$$

Indeed, since $-\log|x-y| \geq 0$ for all $x, y \in [0,1]$, we have $\mathcal{E}(\mu^i) \geq \mathcal{E}(\mu^i|_{B^i})$, and since $\mu^i|_{B^i}/\mu^i(B^i)$ is a probability measure, we have

$$-\log\operatorname{cap} B^i \leq \mathcal{E}(\mu^i|_{B^i}/\mu^i(B^i)) = \frac{\mathcal{E}(\mu^i|_{B^i})}{\mu^i(B^i)^2} \leq \frac{\mathcal{E}(\mu^i)}{\mu^i(B^i)^2} \leq \frac{-\log\operatorname{cap} B^i}{16\mu^i(B^i)^2}.$$

Since ϕ is measure preserving we thus have $\mu^+(B^+ \cup \phi^{-1}(B^-)) \leq 1/2$ and in particular the *good set* $I^+ \setminus (B^+ \cup \phi^{-1}(B^-))$ is non-empty. Let p be a point of this good set. Then there are curves $\gamma^+ \subset S^+$ and $\gamma^- \subset S^-$ joining the top edge of S^+ to p and the bottom edge of S^- to $\phi(p)$ respectively, such that both the ρ^i−lengths of the γ^i are less than $1/2$. Thus the metric $\rho = (\rho^-, \rho^+)$ is not admissible for Γ. It follows that every admissible metric ρ must violate (2.9) and so

$$\mathrm{Area}(\rho) \geq \max_i \mathrm{Area}(\rho^i) > \frac{\pi/8}{16\ \mathcal{E}(\mu^i) + c_0},$$

which proves the proposition. $\square$

Remark 2.4.5. Proposition 2.4.2 can be viewed as a direct consequence of Proposition 2.4.4, since uniformly perfect sets, having positive capacity, support measures of bounded energy, and quasisymmetric maps distort energy of measures in a controlled manner (since they are Hölder continuous).

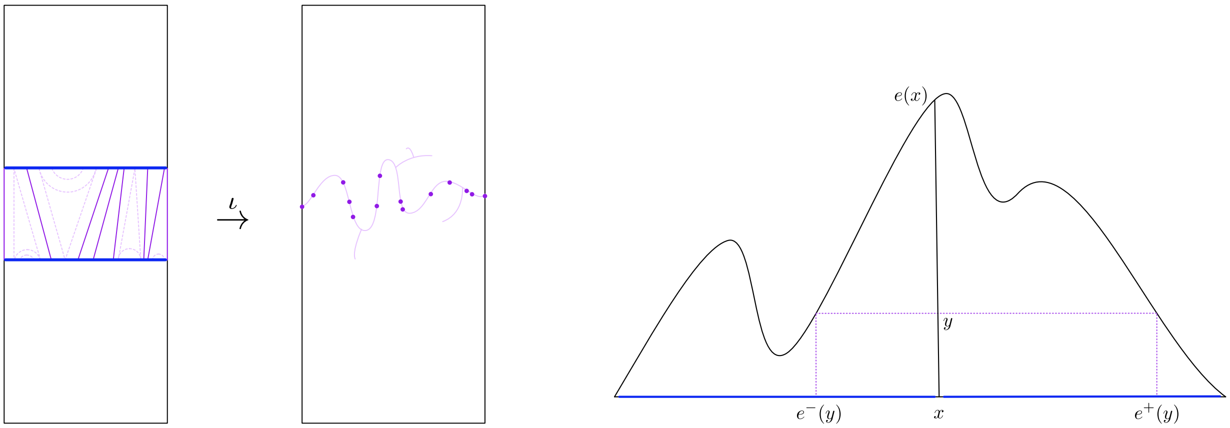
The following lemma explains how Proposition 2.4.4 will be applied to the Brownian lamination, see Chapter 4 and in particular Proposition 4.2.4. Recall that an excursion is a continuous map $e : [0,1] \to [0,\infty)$ which is zero at the endpoints. Every excursion determines a lamination via (1.2).

Lemma 2.4.6. *Fix $x \in (0,1)$ and let $I^- = (0,x) \subset \mathbb{T}$ and let $I^+ = (x,1) \subset \mathbb{T}$. Let $e : [0,1] \to \mathbb{R}^+$ be a (C,α)-Hölder continuous excursion and let $\sim_e$ be the lamination on $\mathbb{T}$ induced by e. There is a gluing pair μ^-, μ^+ for $\sim_e, I^-, I^+$ such that*

$$\max\left(\mathcal{E}(\mu^+), \mathcal{E}(\mu^-)\right) \leq \frac{1}{\alpha}\left(3/2 + \log e(x)^{-1} + \log C\right). \tag{2.11}$$

Proof. Let m be the Lebesgue measure on $[0, e(x)]$ normalized so that $|m| = 1$. Computing the integral (2.2) shows that $\mathcal{E}(m) = \mathcal{E}(m_{[0,1]}) - \log e(x) = 3/2 - \log e(x)$ where $m_{[0,1]}$ is the Lebesgue measure on $[0,1]$.

Define the one-sided inverse functions e^-, e^+ on $[0, e(x)]$ by $e^-(y) = \sup\{t \leq x : e(t) = y\}$ and $e^+(y) = \inf\{t \geq x : e(t) = y\}$, see Figure 2.1b. Let $E^- \subset I^-$ and $E^+ \subset I^+$ be the image of $[0, e(x)]$ under e^- and e^+ respectively. Define the measures μ^+, μ^- on $[0,1]$ by $\mu^+(A) = m(e(A \cap E^+))$ and $\mu^-(A) = m(e(A \cap E^-))$. Since e is (C,α)-Hölder continuous, (2.2) shows that μ^+ and μ^- both have energy bounded above by $\frac{1}{\alpha}\mathcal{E}(m) + \frac{1}{\alpha}\log C = \frac{1}{\alpha}\left(3/2 + \log e(x)^{-1} + \log C\right)$.



(a) Two squares are welded along a common horizontal boundary, via an equivalence relation $\sim$, shown in purple. A measure preserving bijection between subsets of I^+ and I^- is highlighted in dark purple. ι maps the vertices of the rectangle on the left to the vertices of the rectangle on the right. If $\mathrm{dom} f \cup \mathrm{im} f$ is 'large' and f is 'good', then we can bound the aspect ratio of the resulting rectangle.

(b) If $e : [0, 1] \to \mathbb{R}^+$ is a Hölder continuous excursion then we for any fixed x we can construct a measure μ^+ on $[x, 1]$ by pushing forward Lebesgue measure on $[0, e(x)]$ via the inverse e^+ of e that maps y to the first time that e hits y after x. We use a similar construction to get a measure μ^- on $[0, x]$. The energies of μ^- and μ^+ are bounded by a function of C, α and $e(x)$.

Figure 2.1: Illustrations for Proposition 2.4.4 and Proposition 2.4.6.

Define the bijection $\phi : E^- \to E^+$ by $\phi(t) = e^+(e(t))$. Notice that $\min_{t \le s \le \phi(t)} e(s) = e(t) = e(\phi(t))$, so by definition of $\sim_e$ we have $t \sim_e \phi(t)$. We also have $\mu^-(A) = m(e(A \cap E^-)) = m(e(e^-(e(A \cap E^-)))) = m(e\phi^{-1}(A \cap E^-)) = \mu^+(\phi^{-1}(A))$ for all Borel $A \subset [0, 1]$. Thus μ^- and μ^+ is a gluing pair for $\sim_e, I^-, I^+$. $\square$

Finally, we show that the result of Proposition 2.4.4 can be 'chained together' to get an estimate for weldings of multiple rectangles.

Proposition 2.4.7. *For $i = 1, 2, \ldots, m$ consider the square $S_i = [0,1] \times [2i-2, 2i-1]$. Let $\sim_i$ be an equivalence relation identifying points on the edge $[0,1] \times \{2i-1\}$ to the edge $[0,1] \times \{2i\}$. Let Γ be the family of paths $(\gamma_1, \ldots, \gamma_m)$ on $\sqcup S_i$ such that*

- *Each γ_i joins the top edge of S_i to the bottom edge of S_i.*

- *For $i = 1, \ldots, m-1$, the top endpoint of γ_i is identified via $\sim_i$ to the bottom endpoint of γ_{i+1}.*

Then

$$\mathrm{Mod}(\Gamma)^{-1} \leq m \left(C_0 \cdot \max_i \inf_{\mu^-, \mu^+} (\mathcal{E}(\mu^-), \mathcal{E}(\mu^+)) \wedge 1 \right)$$

where the infimum is over all gluing pairs μ^-, μ^+ for $\sim_i$.

Proof. Let $\rho = (\rho_1, \ldots, \rho_m)$ be a metric on $\sqcup_i S_i$ and suppose that

$$\mathrm{Area}(\rho)^{-1} \geq C_0 \max_i \inf_{\mu^-, \mu^+} (\mathcal{E}(\mu^-), \mathcal{E}(\mu^+)) \wedge 1$$

where C_0 is the constant of Proposition 2.4.4. We will construct a path in Γ that has ρ-length bounded above by $C_0 m$, and this will prove the theorem.

For $i = 1, \ldots, m-1$, applying Proposition 2.4.4 to the pair of squares S_i, S_{i+1} and the relation $\sim_i$ shows that there exists a pair of paths $\gamma_{i,i+1} \in S_i$ and $\gamma'_{i,i+1} \in S_{i+1}$ such that

- $\ell_\rho(\gamma_{i,i+1}) + \ell_\rho(\gamma'_{i,i+1}) \leq 1$

- $\gamma_{i,i+1}$ joins the bottom edge of S_i to the top edge of S_i

- $\gamma'_{i,i+1}$ joins the bottom edge of S_{i+1} to the top edge of S_{i+1}

- The top endpoint of $\gamma_{i,i+1}$ is identified to the bottom endpoint of $\gamma'_{i,i+1}$.

See Figure 2.2.

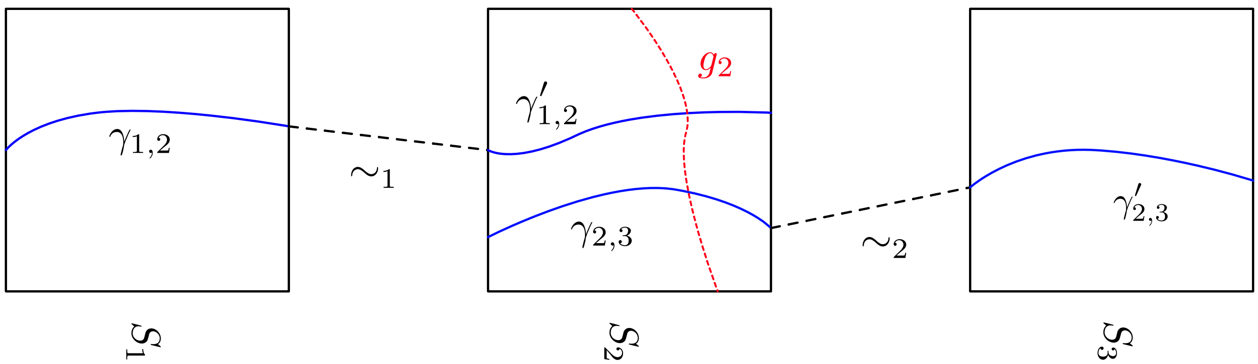


Figure 2.2: Bounding the modulus of multiple squares welded together, Proposition 2.4.7. We use Proposition 2.4.4 to construct paths (blue) joining the boundaries of pairs of squares. Then we use paths (red) joining the other sides of the squares, to find a single path through all the squares.

On the other hand, the modulus of the path family joining the left side of S_i to the right side of S_i is 1, see Example 2.3.1. Therefore for each $i = 2, \ldots, m-1$ there exists a curve g_i joining the left side of S_i to the right side of S_i, with ρ-length bounded above by 1.

Then the union $\bigcup_i \gamma_{i,i+1} \cup \gamma'_{i,i+1} \cup \bigcup_i g_i$ contains a path in Γ with ρ-length bounded by $(m-1)+(m-2) = 2m-3$ (follow that $\gamma_{i,i+1}$ path, then the $\gamma'_{i,i+1}$ path until you intersect the g_{i+1} path, follow that until you hit the $\gamma_{i+1,i+2}$ path, and so on). This proves the desired bound on $\mathrm{Mod}(\Gamma)$. □

2.5 Chains of rectangles and conformal annuli

In this section we will set up some notation for describing the annuli that we will use to control the modulus of continuity of conformal welding solutions. Then we write down sets of conditions that allow control of the modulus of these annuli. Fix a lamination $\sim$ on $\mathbb{T}$.

A *chain link* is pair of disjoint closed intervals $J^- = [a^-, b^-], J^+ = [a^+, b^+] \subset \mathbb{T}$ such that some point of J^- is equivalent to some point of J^+. Often (but not necessarily) the endpoints will be equivalent, $a^- \sim b^+$ and $a^+ \sim b^-$, which explains some of our terminology. For $m \geq 1$,

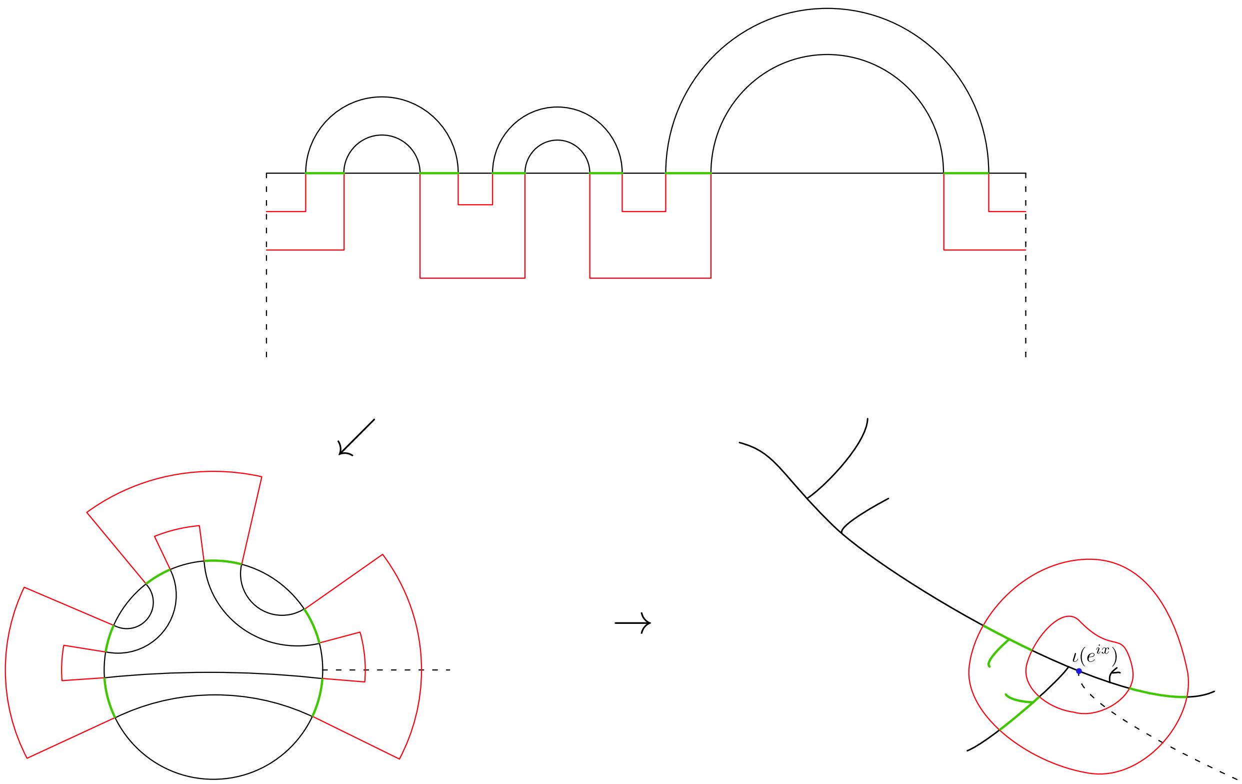


Figure 2.3: Top: The intervals of a 3-chain (green) represented in $\mathbb{R}$. The first two intervals (from the left) form a chain link, the third and fourth intervals form a chain link, and the last two intervals form a chain link. The arcs above the real line represent chords in $\sim$. The regions below the real line bounded by the squares and the real axis form $\mathcal{A}_{\mathbb{H}}(\mathcal{C})$. Bottom left: The lower half plane in the upper figure is mapped to the exterior of the unit disk $\mathbb{D}^*$. Again, the arcs in the disk represent chords of $\sim$. The regions bounded by the curves in $\mathbb{D}^*$ and the unit circle form $\mathcal{A}_{\mathbb{D}^*}(\mathcal{C})$. Bottom right: The image of $\overline{\mathbb{D}^*}$ under the solution to the conformal welding ι. We see that image of $\mathcal{A}_{\mathbb{D}^*}(\mathcal{C})$ from the bottom left picture becomes a single topological annulus around $\iota(e^{ix})$ in $\mathbb{C}$, which we call $\mathcal{A}(\mathcal{C})$. The modulus of the chain is defined to be the modulus of this annulus.

an $m-chain$ $\mathcal{C}$ is a collection of m mutually disjoint chain links J_i^-, J_i^+ such that the intervals are in cyclic order on the circle, $a_1^- < b_1^- < a_1^+ < b_1^+ < a_2^- < ... < a_m^+$.

For $m \geq 2$, the *exterior* of an m-chain is the union of the arcs that lie 'in between' each chain link: $\text{Exterior}(\mathcal{C}) = \cup_{i=1}^m J_i^\circ$ where J_i° is the component of $\mathbb{T} \backslash \{J_i^- \cup J_i^+\}$ which does not contain any other chain links in the chain. The *interior* of an m-chain is the union of arcs $\mathbb{T} \backslash (\text{Exterior}(\mathcal{C}) \cup \mathcal{C})$. These definitions do not make sense when $m = 1$, since $\mathbb{T} \backslash (J_1^- \cup J_1^+)$ consists of exactly two components, neither of which contains any chain links in $\mathcal{C}$. For this reason, we assume that 1-chains come with extra data specifying which components of $\mathbb{T} \backslash (J_1^- \cup J_1^+)$ should be considered the interior and exterior respectively.

Let $\mathcal{C}$ be a m chain. For each $1 \leq i \leq m$, we can associate a topological rectangle $D_i = D_i(\mathcal{C})$ in $\mathbb{D}^*$ as follows. Let I_i be the component of Interior($\mathcal{C}$) lying in between J_i^+ and J_{i+1}^-. Let $\text{Log} : \mathbb{D}^* \backslash L \to \mathbb{C}$ be a branch of the logarithm where the branch cut $L = e^{i\theta}[0, \infty)$ is taken to be any ray from the origin that does not pass through the arcs $J_i^+ \cup I_i \cup J_{i+1}^-$. Then the map $\varphi(z) = \frac{1}{2\pi i}\text{Log} z$ transforms $\mathbb{D}^* \backslash L$ to the vertical half-strip $S_\theta := (\theta, \theta+1) \times (-\infty, 0)$ and maps the arcs J_i^+, I_i and J_{i+1}^- onto intervals in $\mathbb{R}$. Let $A_i = \text{Square}(\varphi(J_i^+ \cup I_i \cup J_{i+1}^-)) \backslash \overline{\text{Square}(\varphi(I_i))}$ where, for I an interval in $\mathbb{R}$, $\text{Square}(I) = I^\circ \times (0, -|I|)$ is the open square in the half-strip S_θ with upper edge I°.

Then A_i can be considered a topological rectangle by taking $\varphi(J_i^+)$ and $\varphi(J_{i+1}^-)$ as the two marked arcs, and D_i is defined to be the image of A_i under the inverse of φ, which is $z \mapsto e^{2\pi i z}$.

The point of this construction is that if $\iota(\overline{\mathbb{D}^*}) = \mathbb{C}$, then $A(\mathcal{C}) := \iota(\cup_i D_i(\mathcal{C}))$ is a topological annulus in $\mathbb{C}$, which we can use with Lemma 2.3.4. Moreover, the bounded component of $\mathbb{C} \backslash A(\mathcal{C})$ contains $\iota(\text{Interior}(\mathcal{C}))$ and the unbounded component of $\mathbb{C} \backslash A(\mathcal{C})$ contains $\iota(\text{Exterior}(\mathcal{C}))$, which explains the terminology.

To use this construction with Lemma 2.3.4, we need to control the modulus of the welded annulus $A(\mathcal{C})$. The bound of Proposition 2.4.4 (more precisely, its proof) gives a way to do this. We see that there are three conditions that need to be satisfied by the chain $\mathcal{C}$ so that the modulus of $A(\mathcal{C})$ is controlled.

Condition 1: All topological rectangles (D_i, J_i^-, J_i^+) have bounded modulus. This is easily seen to be equivalent to requiring an estimate of the form

$$|a_{i+1}^- - b_i^+| \lesssim_L |J_i^+| \asymp_L |J_{i+1}^-|. \tag{2.12}$$

Condition 2a): The lamination is $L-$*thick* between J_i^- and J_i^+, meaning that there exists gluing pairs μ_i^-, μ_i^+ for $\sim, J_i^-, J_i^+$ such that

$$\max\left(\mathcal{E}(\hat{\mu_i^-}), \mathcal{E}(\hat{\mu_i^+})\right) \leq L, \tag{2.13}$$

where $\hat{\mu}$ denotes the probability measure on $[0,1]$ obtained from μ by linear scaling.

Note that the preceding condition is implied by the following stronger condition, by Remark 2.4.5.

Condition 2b): The lamination is $L-$quasisymmetrically thick (see the discussion after Proposition 2.4.2) between J_i^- and J_i^+.

Condition 3: The number of links of the chain is bounded:

$$m \leq L. \tag{2.14}$$

We say that a chain satisfying conditions 1, 2a) and 3 is a *L-good chain.* A chain satisfying 1, 2b) and 3 is a *L-qs-good chain.* If J_i^- and J_i^+ satisfies condition 2b), we say that J_i^- and J_i^+ are *qs-glued.* As explained in Remark 2.4.5, a qs-good chain is automatically a good chain.

Theorem 2.5.1. *If $\mathcal{C}$ is a L-good chain,*

$$\mathrm{Mod}(A(\mathcal{C})) \gtrsim_L 1. \tag{2.15}$$

Proof. The proof is essentially the same as the proof Proposition 2.4.7. Note that we only need one direction of the inequality in (2.12). □

The criterion above motivates the following definition.

Definition 2.5.2. Fix $L > 1$. A number $0 < r < 1$ is a *(qs-)good scale* for a point $x \in \mathbb{T}$, if there is an L-(qs-)good chain $\mathcal{C}$ with $x \in \mathrm{Interior}(\mathcal{C})$, and $L^{-1}r \leq |J_1^+| \leq Lr$.

We conclude this section by stating an analogue of the 'Fundamental Theorem of Conformal Welding', Theorem 1.1.4, for dendrites. A *Gehring tree* $K \subset \mathbb{C}$ is a connected, locally connected, compact set with at least two points such that $\mathbb{C} \backslash K$ is a John domain.

Theorem 2.5.3 ([44, Theorem 1.1]). *Let $\sim$ be a lamination for which $\overline{\mathbb{D}^*}/\sim$ is a topological sphere. If there are C, N and $\varepsilon > 0$ such that for every $x \in \mathbb{T}$, the set of good scales has lower density $\geq \varepsilon$,*

$$\frac{|\{k \leq n : 2^{-k} \in G(x, C, N)\}|}{n} \geq \varepsilon$$

for all n, then $\sim$ is the lamination of a Hölder-tree. If every scale r is good, then $\sim$ is the lamination of a Gehring tree.

Proof Sketch. Suppose $\sim$ satisfies the hypotheses. For the sake of exposition, first suppose that $\sim$ has a conformal welding solution $\iota : \mathbb{D}^* \to \mathbb{C}$. Now fix $x \in \mathbb{T}$. If all scales $0 < r < 1$ are good for x, then for fixed $\varepsilon > 0$, there are $\asymp_L \log \varepsilon^{-1}$ L-good chains $\mathcal{C}$ which surround $B(x, \epsilon)$. From Theorem 2.5.1, this implies that there are $\asymp_L \log \varepsilon^{-1}$ annuli of modulus greater than $\delta = \delta(L)$ surrounding $\iota(B(x, \epsilon))$. Then Lemma 2.3.4 implies that $\operatorname{diam} \iota(B(x, \epsilon)) \lesssim \epsilon^{\alpha(L)}$ where the exponent $0 < \alpha(L) < 1$ depends on L. It follows that the welding ι is Hölder continuous, if it exists.

To actually show the existence of ι, the idea is to consider carefully chosen approximate solutions ι_n that partially solve the welding problem $\sim$. The argument above can be modified to show that the approximations ι_n are uniformly Hölder continuous, so we can pass to subsequential limits. The existence of the welding follows from showing that the subsequential limit solves the welding problem. The Gehring property of the solution can also be deduced from the modulus estimates. □

Combining Proposition 1.1.3 with P. Jones' removability theorem [35] for John domains, or its generalization the Jones-Smirnov removability theorem [36] for Hölder domains, shows that the solution to the welding of Theorem 2.5.3 is unique up to a linear map (see Proposition 1.1.3).

Chapter 3

JULIA SETS OF SEMIHYPERBOLIC QUADRATIC POLYNOMIALS

3.1 Introduction

This chapter is concerned with welding relations arising in the study of the dynamics of quadratic polynomials. First we review the basic definitions in this area, see [16] and [47] for more detailed introductions. Let p be a complex quadratic polynomial. By conjugation via linear maps, we may assume that p is of the form $p_c(z) = z^2 + c$, for some $c \in \mathbb{C}$.

A *periodic point* is a point $z \in \mathbb{C}$ such that $p^n(z) = z$ for some $n \geq 1$. A point z is said to be *pre-periodic* if some iterate $p^t(z)$ is a periodic point. A *strictly pre-periodic* point is a point which is pre-periodic but not periodic. A peroidic point z is said to be *parabolic* if $(p^n)'(z)$ is a root of unity. The *filled Julia set* is the set of points that stays bounded under iteration by p,

$$K_c = \{z : p_c^n(z) \text{ bounded in } n\},$$

and the *Julia set* is the boundary of this set, $J_c = \partial K_c$. The *Fatou set* is the complement of the Julia set. The *basin of attraction to infinity* is the complement of the filled Julia set. It turns out that geometric properties of the filled Julia set are closely related to behaviour of the critical point ($z = 0$) under iteration by p_c.

A simple example is that $0 \in K_c$ if and only if K_c is connected, [16, Theorem 4.1].

The classification of Fatou components, initiated by Fatou and Julia in the 1900s and completed by Sullivan in 1985 ([16, Theorem 2.1] and [56]) implies that if the critical point, 0, of $p_c(z) = z^2 + c$ is strictly preperiodic, then $K_c = J_c$ and in particular J_c is a dendrite.

Example 3.1.1. Let $c = -2$, then the critical point $z = 0$ is strictly preperiodic under iteration by $p_c : 0 \mapsto -2 \mapsto 2 \mapsto 2$, and by the discussion above, J_{-2} must be a dendrite.

This particular example is simple enough that we can see this directly. The map $p_{-2} : z \mapsto z^2 - 2 : \mathbb{C}\backslash[-2,2] \to \mathbb{C}\backslash[-2,2]$ is conjugate to $p_0 : z \mapsto z^2 : \mathbb{D}^* \to \mathbb{D}^*$ via the Joukowsky map $\varphi : z \mapsto z + \frac{1}{z} : \mathbb{D}^* \to \mathbb{C}\backslash[-2,2]$. Indeed, direct calculation shows that $\varphi \circ p_0 \circ \varphi^{-1} = p_{-2}$ on $\mathbb{C}$. Since the basin of attraction to infinity for p_0 is obviously $\mathbb{D}^*$, it follows that $J_{-2} = [-2,2]$, which is indeed a dendrite. Note that φ is the unique normalized welding solution to the lamination $x \sim -x$ on $\mathbb{T}$, since $\varphi(e^{i2\pi x}) = \varphi(e^{-i2\pi x})$.

When K_c is connected and locally connected, the conjugacy above always holds. Let $\varphi_c : \mathbb{D}^* \to \mathbb{C}\backslash J_c$ be the conformal map, normalized so that $\varphi_c(z) = \lambda z + O(1)$ for $\lambda > 0$ (the proof below will show that this forces $\lambda = 1$). It will be convenient to introduce the *Caratheodory loop* $\gamma(x) = \varphi_c(e^{2\pi i x})$ and the doubling map $h(x) = 2x$ for $x \in \mathbb{T}$. Here, we identify $\mathbb{T}$ with the quotient $[0,1]/\sim$, where $\sim$ identifies 0 and 1.

Proposition 3.1.2. *For $x \in \mathbb{T}$,*

$$p_c \circ \gamma(x) = \gamma \circ h(x). \tag{3.1}$$

Proof. The map $\varphi_c^{-1} \circ p_c \circ \varphi_c : \mathbb{D}^* \to \mathbb{D}^*$ is proper of degree 2 and fixes ∞, so $\varphi_c^{-1} \circ p_c \circ \varphi_c(z) = \mu z^2$ for some $|\mu| = 1$. On the other hand, the normalization assumption $\varphi_c(z) = \lambda z + O(1)$ (where $\lambda > 0$) implies that $\varphi_c^{-1} \circ p_c \circ \varphi_c(z) = \lambda z^2 + O(z)$ for large z, so we conclude that $\mu = 1$ (and furthermore $\lambda = 1$). The result then follows from the definition of γ and h. $\square$

This observation allows us to describe the welding relation $\sim$ associated to J_c in a purely combinatorial manner.

Example 3.1.3. Let $c = i$, then the critical point $z = 0$ is strictly preperiodic under iteration by $p_c : 0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i$, and by the discussion above, J_{-i} must be a dendrite.

The Julia set of this polynomial is far more complicated than that of the previous example, see Figure 1.7. However, we can use the semiconjugacy of Proposition 3.1.2 to determine the associated welding relation $\sim$ on $\mathbb{T}$.

Let α be an *external angle* of the critical value $c = i$, meaning $\gamma(\alpha) = i$. Since $p_i^2(i) = p_i^3(i)$, the semiconjugacy (3.1) implies $2^2\alpha = 2^3\alpha \pmod 1$. It follows that $\alpha = k/6$ for some integer k. We know that $\alpha \neq 0$, otherwise this would imply that i is a fixed point of p_i. Similarly, $\alpha \neq \frac{2}{6}, \frac{4}{6}$ because this would imply that i is periodic under iteration by p_i. The only remaining possibilities are $\alpha = 1/6$ or $\alpha = 5/6$. The latter may be eliminated by using Douady-Hubbard correspondence between parameter rays for the Mandelbrot set and dynamical rays for the Julia set, [21, Theorem 8.2].

This fact leads to a nice description the welding $\sim$. The semiconjugacy implies $\gamma \circ h^{-1}(t) \in p_c^{-1}(\gamma(t))$ for $t \in \mathbb{T}$. Applying this with $t = 1/6$ yields $\gamma(1/12) = \gamma(7/12) = 0$, because 0 is the only element of $p_c^{-1}(i)$. In particular, $1/12 \sim 7/12$.

By the same reasoning, we get that $\{\gamma(1/24), \gamma(13/24), \gamma(7/24), \gamma(19/24)\} \in p_c^{-1}(0) = \{(-i)^{1/2}, -(-i)^{1/2}\}$. From topological considerations, the welding relation $\sim$ must be *flat* [57, Proposition II.3.3], meaning that if $x \sim y$ and $z \sim w$ and the chord $[x, y]$ crosses $[z, w]$, then $x \sim y \sim z \sim w$.

Using this, we can deduce that $\gamma(1/24) = \gamma(7/24)$ and $\gamma(19/24) = \gamma(13/24)$, and that these two points are distinct.

Continuing inductively, we get an infinite set of pairs in $\sim$. It can be shown that taking the topological closure of these pairs in $\mathbb{T} \times \mathbb{T}$ recovers $\sim$ itself, see [8, Theorem 1]

In the previous example we saw that the parameter $\alpha = 1/6$ generates a lamination $\sim$ via backwards iteration. In Sections 3.2.1 and 3.2.2 we generalize this. For each $\alpha \in \mathbb{T}$ there is an associated equivalence relation $\approx_\alpha$, defined without reference to complex analysis. When the Julia set J_c of a quadratic polynomial p_c is a locally connected dendrite, the lamination describing its welding is equal to $\approx_\alpha$, where $\alpha \in \mathbb{T}$ is the landing angle of the critical value c.

Another example where the dynamical behaviour of the critical point influences the geometry of the Julia set is captured by the notion of semihyperbolicity, which was introduced for polynomials in [17]. It was shown (among other things) that a polynomial p is semihyperbolic if and only if either of the following equivalent conditions hold. In this chapter we will take the second condition as our definition of semihyperbolicity.

Theorem 3.1.4 ([17, Theorem 1.1])**.** *The following conditions are equivalent.*

- *p is semihyperbolic.*
- *p has no parabolic periodic points and $w \notin \overline{\bigcup_{t\geq 1} p^t(w)}$ for all points w such that $p'(w) = 0$ (critical points).*
- *The basin of attraction to ∞ is a John domain.*

Note that if p_c is strictly preperiodic, then it is semihyperbolic.

In this chapter, we introduce a combinatorial condition on the parameter α called *combinatorial semihyperbolicity*, (see Definition 3.2.5). We show directly that if α is combinatorially semihyperbolic , then the lamination $\approx_\alpha$ associated to α satisfies the hypotheses of Theorem 2.5.3, see also Definition 2.5.2.

Theorem 3.1.5. *Suppose $\alpha \in \mathbb{T}$ is combinatorially semihyperbolic and consider the lamination $\approx_\alpha$ associated to α. Then there exists L such that for all $x \in \mathbb{T}$, for all $0 < r < 1$, r is a $L-$qs-good scale for x.*

In fact, our construction will show that we can take the uniformly perfect sets (in the definition of qs-good scales) to be linear Cantor sets, and we can take the quasisymmetric maps to be linear maps.

We also prove that combinatorial semihyperbolicity is equivalent to semihyperbolicity.

Theorem 3.1.6. *Suppose $c \in \mathbb{C}$ is a parameter for which J_c is a dendrite. Suppose c is semihyperbolic. Let $\alpha \in \mathbb{T}$ be a landing angle of the critical value c. Then α is combinatorially semihyperbolic.*

Conversely, if α is combinatorially semihyperbolic, there exists $c \in \mathbb{C}$ for which J_c is a dendrite and c is semihyperbolic, and the external ray at angle α lands at c, and $\approx_\alpha = \sim_c$.

The proof of this Theorem is at the end of this chapter, in Section 3.4. Everything preceding that section is purely combinatorial.

As a corollary of Theorems 2.5.3, 3.1.5 and 3.1.6, we get a new proof of Theorem 3.1.4 in the case where p is a quadratic polynomial with dendritic Julia set.

Notation In this chapter we make the identification $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, and for $a, b \in \mathbb{T}$ we write (a, b) to mean the counterclockwise open arc from a to b, and similarly for $[a, b]$. We define the doubling map $h : \mathbb{T} \to \mathbb{T}$ by $h(x) = 2x \bmod 1$. If $I = (a, b) \subset \mathbb{T}$ or $I = [a, b] \subset \mathbb{T}$ is an open or closed interval, let $|I| = b - a$ denote its length, normalized so that $|\mathbb{T}| = 1$.

If $a, b, c, d \in \mathbb{T}$ are distinct, we say that $\{a, b\}$ *crosses* $\{c, d\}$ if the chord joining a to b intersects (in $\overline{\mathbb{D}}$) the chord from c to d. This is equivalent to saying that if U_1 and U_2 are the two components of $\mathbb{T}\backslash\{a, b\}$, then $\overline{U_1}$ and $\overline{U_2}$ each contain an element of $\{c, d\}$. We sometimes refer to two element subsets $\{a, b\} \subset \mathbb{T}$ as *chords*. If $a \sim b$ for some relation $\sim$ (depending on context), we may refer to $\{a, b\}$ as a *leaf*.

An equivalence relation $\sim$ on $\mathbb{T}$ is *flat* if whenever $\{x, y\}$ and $\{z, w\}$ are crossing leaves, then $x \sim y \sim z \sim w$. A *lamination* is a flat equivalence relation on $\mathbb{T}$.[1]

A^* is the set of finite words on some alphabet set A, and A^∞ is the set of infinite words. If $g \in A^*$ is a finite word, we write $|g|$ to denote the number of letters in g. If, in addition, $h \in A^* \cup A^\infty$ is a finite or infinite word, gh denotes the concatenation. If k is in a positive integer, g^k denotes the k-fold concatenation and g^∞ denotes the periodic infinite word $ggg\ldots$. For integer $m \geq 0$, g_m denotes the mth letter of g, and $g|_m$ denotes the subword of g of the first m letters $g_0 g_1 g_2 \ldots g_{m-1} \in A^m$.

If $\sim$ is an equivalence relation on $\mathbb{T}$ and $x \in \mathbb{T}$, we write $[x]$ or $[x]_\sim$ to denote the equivalence class of x as a subset of $\mathbb{T}$.

To reduce clutter, we will drop subscripts, superscripts, and parentheses for function arguments when they are clear from context.

[1] In the literature, for instance [57], a *lamination* refers to a collection of noncrossing chords of the circle. The two usages are clearly related, but in this chapter a *lamination* will always be an equivalence relation.

3.2 Description of the lamination

We provide a mostly self contained review of the minimal and dynamical α-equivalences $\sim_\alpha$ and $\approx_\alpha$ in Section 3.2.1 and 3.2.2. See [57] and [8] for more detailed presentations. These equivalences are equal in our setting (see Theorem 3.2.4), and they provide a family of conformal welding problems. In Section 3.2.3 we introduce the notion of M-closeness which provides a simple characterization of the topology of quotient space $\mathbb{T}/\approx_\alpha$.

Our construction will use some more detailed results on the relationship between $\sim_\alpha$ and $\approx_\alpha$, and this is the content of Section 3.2.4.

3.2.1 Invariant Laminations and the Minimal α-Equivalence $\sim_\alpha$

Fix $\alpha \in \mathbb{T}\backslash\{0\}$, and assume that α is not periodic under iteration by h (we will soon see that this follows from the combinatorial semihyperbolicity condition defined in Section 3.1). The preimages of α under the angle doubling map h are $*_1 := \alpha/2$ and $*_2 := \alpha/2 + 1/2$. These two points divide the circle $\mathbb{T}$ into two semi-circular arcs, $(*_1, *_2)$ and $(*_2, *_1)$. See Figure 3.1a. Let L be the semicircle containing α and let R be the other semicircle.

Each $x \in \mathbb{T}$ has two preimages under h. For $x \neq \alpha$, these pre-images $x/2$ and $x/2 + 1/2$ lie in either the semicircle $(*_1, *_2)$ or in the semicircle $(*_2, *_1)$. We let $\widetilde{L}x$ be the pre-image that lies in L and let $\widetilde{R}x := \widetilde{L}x + 1/2$ be the other pre-image of x, lying in R. Let $\sim$ be an equivalence relation on $\mathbb{T}$, with a marked point $\alpha \in \mathbb{T}$. Here are some properties that $\sim$ may have:

1. Forward invariant: $x \sim y \implies h(x) \sim h(y)$

2. Backward Invariant: For $x, y \neq \alpha$, we have $x \sim y \implies \widetilde{L}x \sim \widetilde{L}y$ and $\widetilde{R}x \sim \widetilde{R}y$. If $x \neq \alpha$ and $x \sim \alpha$, then $\widetilde{L}x \sim *_1$ and $\widetilde{R}x \sim *_1$.

3. Closed: If $x_n \to x$ and $y_n \to y$ and $x_n \sim y_n$ for all n then $x \sim y$.

Following [57], any equivalence relation satisfying properties 1) and 2) above is said to be *invariant*.

The *minimal α-equivalence* is the minimal closed invariant equivalence relation $\sim_\alpha$ for which $*_1 \sim_\alpha *_2$. It is a lamination, see [57, Proposition II.4.5].

The following general observation about invariant relations will be useful throughout the rest of the chapter, and is easily proved by induction.

Lemma 3.2.1. *Fix $\alpha \in \mathbb{T}$ and suppose $\sim$ is a forward and backward invariant equivalence relation with respect to α. Then for every $x \in \mathbb{T}$ and $t \geq 0$, we have $h^t([x]_\sim) = [h^t(x)]_\sim$.*

Backwards invariance of $\sim_\alpha$ allows us to construct some chords of $\sim_\alpha$ concretely. Recall that $\widetilde{L}$ and $\widetilde{R}$ are the continuous inverse branches of h on $\mathbb{T}\backslash\{\alpha\}$. Every finite word $g \in \{L, R\}^*$ encodes a composition of such mappings, where we use the usual right to left ordering convention for function composition. Denote this mapping by $\widetilde{g}$.

Since each function in the composition $\widetilde{g}$ is only well defined away from α, the domain of $\widetilde{g}$ is $\mathbb{T}\backslash A_g$ where A_g is some subset of the *postcritical set* $P_\alpha := \{h^t\alpha : 0 \leq t \leq |g|\}$. In fact, it is easy to see that

$$A_g = \{x \in \mathbb{T} : \widetilde{\sigma^t g}(x) = \alpha \text{ for some } 1 \leq t \leq |g|\} \tag{3.2}$$

where we recall that σ is the left shift operator on words. Notice that the derivative of $\widetilde{g}$ is $2^{-|g|}$ and $\widetilde{g}$ is linear on each component of $\mathbb{T}\backslash A_g$.

Since α is not periodic, $*_1$ and $*_2$ are not in the postcritical set P_α. Therefore $\widetilde{g}*_1$ and $\widetilde{g}*_2$ are well defined for every choice of g. The following observation follows immediately from backwards invariance.

Proposition 3.2.2. *For all $g \in \{L, R\}^*$, we have $\widetilde{g}*_1 \sim_\alpha \widetilde{g}*_2$.*

If $\widetilde{g}\{*_1, *_2\}$ is a leaf of the minimal equivalence, and $\widetilde{g}$ is finite, we say that $|g|$ is the *depth* of that leaf. The choice of $\widetilde{g}$ used to represent the leaf is unique because α is not periodic, so this is well defined.

3.2.2 Itineraries and the Dynamical α-Equivalence $\approx_\alpha$

In the course of our construction we will need the following alternative description of the lamination $\sim_\alpha$, in terms of itineraries. Every point in $\mathbb{T}$ lies in either L, R or $\{*_1, *_2\}$. See Figure 3.1a. For $x \in \mathbb{T}$, the *itinerary* $I^\alpha(x) \in \{L, R, \star\}^\infty$ is an infinite sequence on a three letter alphabet, which keeps track of which half of the circle the iterates of x lie in. It is defined as follows:

$$I^\alpha(x)_n = \begin{cases} L & \text{if } h^n x \in L \\ R & \text{if } h^n x \in R \\ \star & \text{if } h^n x \in \{*_1, *_2\}. \end{cases} \qquad \text{for } n \geq 0.$$

It follows immediately from the definitions that $h : \mathbb{T} \to \mathbb{T}$ is semiconjugate to the left shift $\sigma : \{L, R, \star\}^\infty \to \{L, R, \star\}^\infty$, which maps $\sigma : u_0 u_1 u_2 \cdots \mapsto u_1 u_2 \ldots$. In other words $I^\alpha(hx) = \sigma I^\alpha(x)$. We also have

$$LI^\alpha(x) = I^\alpha(\widetilde{L}x), \qquad RI^\alpha(x) = I^\alpha(\widetilde{R}x), \qquad \text{for } x \in \mathbb{T}\backslash\{\alpha\}. \tag{3.3}$$

The itinerary $I^\alpha(\alpha)$ plays a special role and it is called the *kneading sequence* for α. When α is combinatorially semihyperbolic (see Definition 3.2.5), $I^\alpha(\alpha)$ is not periodic. Since we are only interested in combinatorially semihyperbolic parameters, we will assume that $I^\alpha(\alpha)$ is not periodic for the rest of this chapter (except possibly in Section 3.4). This also implies that α is not periodic under iteration by h.

The *dynamical α-equivalence* $\approx_\alpha$ is the smallest equivalence relation such that points with the same itinerary are identified, where the $\star$ symbol is used as a wildcard when comparing two itineraries.

Formally, this is described as follows. We say that an infinite word $g \in \{L, R, \star\}^\infty$ is *precritical* if it can be written as $g = usI^\alpha(\alpha)$ where $u \in \{L, R\}^*$ is a finite word and $s \in \{L, R, \star\}$. Note that by nonperiodicity of $I^\alpha(\alpha)$, such a decomposition, if it exists, is unique. If g is precritical and $g = usI^\alpha(\alpha)$ as above, then define the infinite words

$g^L = uLI^\alpha(\alpha)$, $g^R = uRI^\alpha(\alpha)$ and $g^\star = u \star I^\alpha(\alpha)$. Then $\approx_\alpha$ is defined as follows:

$$x \approx_\alpha y \iff \begin{cases} I^\alpha(x) = I^\alpha(y), \quad \text{or } I^\alpha(x), I^\alpha(y) \text{ are precritical and} \\ I^\alpha(x)^\star = I^\alpha(y)^\star. \end{cases} \tag{3.4}$$

Note that $I^\alpha(x)^\star = I^\alpha(y)^\star$ iff $I^\alpha(x)^L = I^\alpha(y)^L$ iff $I^\alpha(x)^R = I^\alpha(y)^R$ iff $I^\alpha(y) \in \{I^\alpha(x)^L, I^\alpha(x)^R, I^\alpha(x)^\star\}$. In particular, if $I^\alpha(x)$ is precritical and $y \in \mathbb{T}$, then $x \approx_\alpha y$ iff $I^\alpha(y) \in \{I^\alpha(x)^L, I^\alpha(x)^R, I^\alpha(x)^\star\}$. If $I^\alpha(x)$ is not precritical then the only way that y can be $\approx_\alpha$ equivalent to x is if $I^\alpha(y) = I^\alpha(x)$.

From this we see that $\approx_\alpha$ as defined in (3.4) is an equivalence relation.

From (3.3) we see that $\approx_\alpha$ is forward and backward invariant, in particular Lemma 3.2.1 applies to $\approx_\alpha$ and therefore $h^t[x]_{\approx_\alpha} = [h^t x]_{\approx_\alpha}$ for all $x \in \mathbb{T}$ and $t \geq 0$.

If $I^\alpha(x)$ is precritical then this is equivalent to saying that $h^t x \in [*_1]$ for some $t \geq 0$. By the above this is also equivalent to saying that $h^t[x] = [*_1]$.

We see that $I^\alpha(\alpha)$ is not precritical, because $I^\alpha(\alpha)$ is not periodic. As a consequence, $[\alpha]$ is not periodic, in other words $h^t[\alpha] = [h^t\alpha] \neq [\alpha]$ for $t \geq 1$.

This then implies that $h^t[*_1] \neq [*_1]$ for $t \geq 1$. We will use these observations repeatedly throughout the rest of this chapter.

We will also need the following results.

Theorem 3.2.3 ([8, Proposition 6.2])**.** *Suppose $I^\alpha(\alpha)$ is nonperiodic. Then all equivalence classes of $\approx_\alpha$ are finite.*

In our setting, the minimal equivalance $\sim_\alpha$ defined in the previous subsection is the same as the dynamical equivalence $\approx_\alpha$.

Theorem 3.2.4 ([8, Theorem 1])**.** *Suppose $I^\alpha(\alpha)$ is nonperiodic. Then $\sim_\alpha = \approx_\alpha$.*

With the concept of itineraries, we can now state our definition of combinatorial semihyperbolicity. It is clearly inspired by Theorem 3.1.4 and the characterization of the topology of $\mathbb{T}/\approx_\alpha$ (Proposition 3.2.7). See also the proof of Theorem 3.1.6.

Definition 3.2.5. We say that $\alpha \in \mathbb{T}$ is *combinatorially semihyperbolic* if there exists $M_\alpha > 0$ integer such that for $t \geq 1$, $I^\alpha(h^t\alpha)|_{M_\alpha} \neq I^\alpha(\alpha)|_{M_\alpha}$.

Clearly, if α is combinatorially semihyperbolic, then its kneading sequence $I^\alpha(\alpha)$ is nonperiodic.

3.2.3 Convergence in $\mathbb{T}/\approx_\alpha$ in terms of itineraries

In this subsection we will develop a useful characterization (Proposition 3.2.7) of the convergent sequences in $\mathbb{T}/\approx_\alpha$ in terms of itineraries of points in the sequence.

The idea is that nearby points should have itineraries that agree on long initial subwords. However, the definition is a little complicated because $\star$ needs to be treated as a wildcard; a good example to keep in mind is the points $*_1+\epsilon$ and $*_1-\epsilon$ for $\epsilon>0$ small. They are close in $\mathbb{T}/\approx_\alpha$, that is $[*_1-\epsilon]$ and $[*_1+\epsilon]$ converge to the same point in $\mathbb{T}/\approx_\alpha$ as $\epsilon\to 0$, but their itineraries differ at the first letter.

Motivated by this example, we say that two length M words $g_1, g_2 \in \{L, R, \star\}^M$ are M-close if there exists a finite word $u \in \bigcup_{k=0}^{M-1}\{L,R\}^j$ of length at most $M-1$ such that $g_1 = us_1v$ and $g_2 = us_2v$, where $s_1, s_2 \in \{L, R, \star\}$, and v is a (possibly empty) initial subword of $I^\alpha(\alpha)$ of length $M-|u|-1$. In particular, if $g_1 = g_2$, then they are M-close.

We can extend this definition to words of length greater than M, including infinite words, by saying that two such words are M-close if their restrictions to the first M letters are M-close. We also say that two points $x, y \in \mathbb{T}$ are M-close if their itineraries are M-close. We use the notation $x \overset{M}{\asymp} y$ and $I^\alpha(x) \overset{M}{\asymp} I^\alpha(y)$ to denote M-closeness.

For each $x\in\mathbb{T}$, we define the M-neighborhood $B_M(x)$ around x to be the set of points in $\mathbb{T}$ which are M-close to x.

First we prove that if y is sufficiently close to x with respect to the standard topology on $\mathbb{T}$, then y is M-close to x.

Lemma 3.2.6. *For each $M>0$ and $x\in\mathbb{T}$, there exists an open neighborhood $U_M(x)$ containing $[x]_{\approx_\alpha}$ such that $y \in U_M(x) \implies x \overset{M}{\asymp} y$. If in addition $I^\alpha(x)$ is not precritical, we can strengthen the conclusion by replacing it with $y \in U_M(x) \implies I^\alpha(x)|_M = I^\alpha(y)|_M$.*

Proof. Recall that $I^\alpha(x)$ is not precritical if and only if $h^t[x] \neq [*_1]$ for $t\geq 0$. From this we see that $I^\alpha(x)$ is not precritical if and only if $h^t[x] = [h^t x]$ never intersects $\{*_1, *_2\}$ for $t \geq 0$.

First suppose that the sets $h^t[x]$ never intersect $\{*_1, *_2\}$ for $t \geq 0$. For each $x' \in [x]$ we can choose an open arc $I_{x'}$, containing x', small enough that $h^t I_{x'} \cap \{*_1, *_2\}$ is empty for $0 \leq t \leq M$. Now, for $0 \leq t \leq M$, we have that $h^t I_{x'}$ and $h^t x'$ lie in the same semicircle (either L or R). Therefore any $y \in I_{x'}$ has the same itinerary as x' for the first M letters. But every $x' \in [x]$ has the same itinerary as x for the first M letters because $I^\alpha(x)$ is not precritical. Therefore every $y \in U_M := \bigcup_{x' \in [x]} I_{x'}$ has the same itinerary as x for the first M letters. Moreover, the first M letters of the common itinerary all lie in $\{L, R\}$.

Now suppose $h^t[x] = [*_1]$ for some $0 \leq t \leq M$. This can happen for at most one value of t because $[*_1]$ is not periodic.

Let T be the unique time such that $h^T[x] = [*_1]$, then we can write $I^\alpha(x)|_M = usv$ where $u \in \{L, R\}^T$, $s \in \{L, R, \star\}$, and $v \in \{L, R\}^{M-T-1}$ is an initial subword of $I^\alpha(\alpha)$. For each $x' \in [x]$, let $I_{x'}$ be an open interval containing x' and small enough such that $h^t I_{x'}$ does not intersect $\{*_1, *_2\}$ for $0 \leq t \leq T-1$ and $T+1 \leq t \leq M$.

The same reasoning as in the previous case shows that if $y \in \bigcup_{x' \in [x]} I_{x'}$ then $I^\alpha(y) = I^a(x)$, except possibly at the index $t = T$. $\square$

The converse to the previous lemma is also true; if x is M-close to y for large M then y is close to $[x]_{\approx_\alpha}$ in $\mathbb{T}$. This gives us the characterization of convergence we wanted.

Proposition 3.2.7. *Suppose x_n is a sequence on $\mathbb{T}$, and suppose $x \in \mathbb{T}$. Then $[x_n] \to [x]$ in the quotient topology of $\mathbb{T}/\approx_\alpha$ iff for all integer M, there exists N_0 such that $n > N_0$ implies $I^\alpha(x_n)$ is M-close to $I^\alpha(x)$.*

Before proving this proposition, we need a few lemmas.

The next lemma is a sort of triangle inequality.

Lemma 3.2.8. *For each $M > 0$ there exists $K > 0$ such that the following holds. If $x \overset{M+K}{\asymp} z$ and $z \overset{M+K}{\asymp} y$ then $x \overset{M}{\asymp} z$.*

Proof. Suppose $x \overset{M+K}{\asymp} z$ and $z \overset{M+K}{\asymp} y$. Then

$$I^\alpha(x)|_{M+K} = uav$$

$$I^\alpha(z)|_{M+K} = ubv \tag{3.5}$$

$$I^\alpha(z)|_{M+K} = u'a'v' \tag{3.6}$$

$$I^\alpha(y)|_{M+K} = u'b'v'$$

where $u, u' \in \{L, R\}^*$ are finite words, $a, b, a', b' \in \{L, R, \star\}$ and v, v' are initial subwords of $I^\alpha(\alpha)$. Suppose first that $|u| \geq M$. If $|u'| \geq M$ too then we are done, because $I^\alpha(x)|_M = u|_M = I^\alpha(z)|_M = u'|_M = I^\alpha(y)|_M$.

On the other hand if $|u'| < M$, we have $I^\alpha(x)|_M = u|_M = I^\alpha(z)|_M = u'a'v''$ where v'' is an initial subword of v', and hence v'' is an initial subword of $I^\alpha(\alpha)$. Also $I^\alpha(y)|_M = u'b'v''$, so $x \overset{M}{\asymp} y$.

The case where $|u'| \geq M$ and $|u| < M$ is then taken care of by the symmetric argument, so now suppose $|u'| < M$ and $|u| < M$. This is the only case where we use the fact that K is large and $I^\alpha(\alpha)$ is nonperiodic. Since $I^\alpha(\alpha)$ is nonperiodic, every shift $\sigma^i I^\alpha(\alpha)$ must eventually disagree with $I^\alpha(\alpha)$ if we look deep enough into the sequence. We choose K to be the largest index we need to observe to find this disagreement, when we restrict to shifts of length less than M. That is,

$$K = \sup_{0 \leq i \leq M} \inf\{T : \sigma^i I^\alpha(\alpha)|_T \neq I^\alpha(\alpha)|_T\}.$$

It suffices to show that $|u| = |u'|$ because then the two different ways of writing $I^\alpha(z)$ forces $u = u'$ and hence $x \overset{M}{\asymp} y$. So suppose for contradiction that, say, $|u| < |u'|$ and write $u' = ucw$ for some $c \in \{L, R\}$ and some (possibly empty) word $w \in \{L, R\}^*$. Note that $0 \leq |w| < |u'| < M$.

Applying the shift $\sigma^{|u|+1}$ to (3.5) and (3.6) gives

$$\sigma^{|u|+1}(I^\alpha(z)|_{M+K}) = v$$

$$\sigma^{|u|+1}(I^\alpha(z)|_{M+K}) = wa'v'$$

which shows $\sigma^{|w|+1}v = v'$. But recall that v and v' are initial subwords of $I^\alpha(\alpha)$, and the lengths of v and v' are at least K, so this means that we actually have $\sigma^{|w|+1}I^\alpha(\alpha)|_K = I^\alpha(\alpha)|_K$. This contradicts the definition of K. □

The following lemma says that the topology induced by the B_M is fine enough to distinguish $\approx_\alpha$ classes.

Lemma 3.2.9. *For each $x \in \mathbb{T}$ and $M > 0$ there exists $K > 0$ such that $\overline{B_{M+K}(x)} \subset B_M(x)$. Furthermore,*

$$[x]_{\approx_\alpha} = \bigcap_M B_M(x) = \bigcap_M \overline{B_M(x)}.$$

Proof. We begin by proving the first equality. If $x \approx_\alpha y$, it follows from the definitions that $x \overset{M}{\asymp} y$ for all M. On the other hand, suppose $x \overset{M}{\asymp} y$ for all M. Let T be the smallest number that $I^\alpha(x)|_{T+1} \neq I^\alpha(y)|_{T+1}$, if $T = \infty$ then we are done so suppose $T < \infty$. Let $u = I^\alpha(x)|_T = I^\alpha(y)|_T$. Then for each $M > T$ we have $x|_M = us_1v$ and $y|_M = us_2v$ where $s_1, s_2 \in \{L, R, \star\}$ and v is an initial subword of $I^\alpha(\alpha)$. Letting $M \to \infty$ shows that $x = us_1I^\alpha(\alpha)$ and $y = us_2I^\alpha(\alpha)$, which means $x \approx_\alpha y$.

Now we turn to the second equality. Fix $M > 0$ and $x \in \mathbb{T}$ and suppose $K > 0$ is large. Suppose $y_i \in B_{M+K}(x)$ is a sequence converging in $\mathbb{T}$ to $y \in \overline{B_{M+K}(x)}$.

Lemma 3.2.6 implies that for sufficiently large i, we have $y_i \overset{M+K}{\asymp} y$. Taking $z = y_i$ in Lemma 3.2.8 implies that $x \overset{M}{\asymp} y$. Thus $\overline{B_{M+K}(x)} \subset B_M(x)$, and this proves the second equality in the statement of the lemma. □

Proposition 3.2.7 now follows from the next lemma.

Lemma 3.2.10. *Suppose $x \in \mathbb{T}$, and let $U \subset \mathbb{T}$ be an open set containing $[x]$. For sufficiently large M, we have $B_M(x) \subset U$.*

Proof. To prove the claim, suppose for contradiction that there is a sequence $y_j \in \mathbb{T} \backslash U$ and $M_j \in \mathbb{R}$ such that $y_j \in B_{M_j}(x)$ and $M_j \to \infty$. By compactness we can assume that $y_j \to y \in \mathbb{T}\backslash U$. We have from Lemma 3.2.9 that for each M, the tail of the sequence (y_j) is contained in $\overline{B_M(x)}$. Therefore $y \in \bigcap_M \overline{B_M(x)}$ and by Lemma 3.2.9 we get $y \in [x]$, which contradicts the fact that U contains $[x]$. $\square$

Proof of Proposition 3.2.7. Assume $[x_n] \to [x]$ in $\mathbb{T}/\approx_\alpha$. Let $M \geq 0$ be arbitrary. By Lemma 3.2.6 there exists an open neighborhood U of $[x]$ in $\mathbb{T}$ such that $y \in U \implies y \overset{M}{\asymp} x$. For sufficiently large n we must have $x_n \in U$, hence $x_n \overset{M}{\asymp} x$. Since M was arbitrary, this proves the 'only if' direction.

For the other direction, let W be an arbitrary open neighborhood of $[x]$ in $\mathbb{T}/\approx_\alpha$, and let U be its preimage under the quotient map $x \mapsto [x]$. Then U is an open set containing $[x] \subset \mathbb{T}$. By Lemma 3.2.10, we have for sufficiently large M that $B_M(x) \subset U$, and this completes the proof. $\square$

3.2.4 Cylinder Sets and Boundary Leaves

The material in this section can also be found in [8, Section 4], but it is a crucial part of the proof, so we provide a self-contained presentation here.

If $g \in \{L, R\}^*$ a finite word, define the *(open) cylinder* $C(g) \subset \mathbb{T}$ to be the set of points whose initial itinerary is equal g, that is $C(g) = \{x \in \mathbb{T} : I^\alpha(x)|_{|g|} = g\}$. These sets are closely related to the M-neighborhoods introduced in the previous section, indeed every M-neighborhood is (up to a finite boundary set) equal to a union of finitely many cylinder sets.

One important part of our proof of Theorem 3.1.5 is in finding leaves in the relation $\sim_\alpha$ which have one endpoint close to a given point x. We use this idea twice, in Lemma 3.3.2 and also in the Lemma 3.3.7. We do this by considering cylinder sets containing x.

1. By the results of the previous section, in particular Lemma 3.2.9, we see that $C(I^\alpha(x)|_n)$ converges to $[x]$ as $n \to \infty$. Actually this only makes sense for x that are not precritical, since we have not defined the cylinder sets for words that contain the symbol $\star$. Proposition 3.2.11 deals with this issue.

2. On the other hand, every boundary chord of a cylinder $C(g)$ is actually a leaf of the lamination (Proposition 3.2.12).

The combination of these results allows us to construct the desired approximations for every x. The rest of this section contains the proofs of these two statements, and the approximation result is summarized in Proposition 3.2.13.

Recall from Section 3.2.2 that an itinerary $g \in \{L, R, \star\}^\infty$ is said to be precritical if it can be written in the form $g = usI^\alpha(\alpha)$ where $u \in \{L, R\}^*$ and $s \in \{L, R, \star\}$. If g is precritical, $g^L = uLI^\alpha(\alpha)$ and $g^R = uRI^\alpha(\alpha)$ are the words obtained from g by replacing the symbol $\star$ with L and R respectively.

Proposition 3.2.11. *Suppose $x \in \mathbb{T}$ and let $g = I^\alpha(x) \in \{L, R, \star\}^\infty$. First suppose g is not precritical. Then*

$$[x]_{\approx_\alpha} = \bigcap_{n\geq 1} \overline{C(g|_n)} = \bigcap_{n\geq 1} C(g|_n).$$

On the other hand, if g is precritical, then

$$[x]_{\approx_\alpha} = \bigcap_{n\geq 1} \overline{C(g^L|_n)} \cup \bigcap_{n\geq 1} \overline{C(g^R|_n)}. \tag{3.7}$$

Moreover, in this case,

$$\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)} = \widetilde{u}\{*_1, *_2\},$$

where $u \in \{L, R\}^*$ is defined implicitly via $g^\star = usI^\alpha(\alpha)$ for some $s \in \{L, R, \star\}$.

Proof. Consider first the case where g is not precritical. It is clear that $\overline{C(g|_n)} \subset \overline{B_n(x)}$. So by Lemma 3.2.9, the intersection $\bigcap_{n\geq 1} \overline{C(g|_n)}$ is contained inside $[x]_{\approx_\alpha}$ On the other hand, if $y \in [x]_{\approx_\alpha}$, this means that $I^\alpha(y) = I^\alpha(x)$. So $y \in C(g|_n)$ for all n. We have shown that $\bigcap_{n\geq 1} \overline{C(g|_n)} \subset [x]_{\approx_\alpha} \subset \bigcap_{n\geq 1} C(g|_n)$, so we are done.

Now consider the case where g is precritical and write $g = usI^\alpha(\alpha)$. Again we have for all $n \geq 1$ that $C(g^L|_n) \subset B_n(x)$ and $C(g^R|_n) \subset B_n(x)$, therefore $\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cup \bigcap_{n\geq 1} \overline{C(g^R|_n)} \subset [x]_{\approx_\alpha}$.

For the other direction, suppose $y \in [x]_{\approx_\alpha}$. There are three cases to consider.

- If $I^\alpha(y) = uLI^\alpha(\alpha) = g^L$ then $y \in C(g^L|_M)$ for all M.
- If $I^\alpha(y) = uRI^\alpha(\alpha) = g^R$ then $y \in C(g^R|_M)$ for all M.
- It remains to check the case $I^\alpha(y) = u \star I^\alpha(\alpha)$. If this is the case, then $y \in \{\widetilde{u}*_1, \widetilde{u}*_2\}$. Assume that $y = \widetilde{u}*_1$, the other case $y = \widetilde{u}*_2$ is similar. We claim that $y = \widetilde{u}*_1 \in \overline{C(g^L|_M)}$ for all M. To see this, observe that $*_1 + \epsilon$ is in L, for all small $\epsilon > 0$. Also $h(*_1+\epsilon)$ is close to α, so for sufficiently small ϵ we have that $I^\alpha(*_1+\epsilon)|_M = LI^\alpha(\alpha)|_M$, see Lemma 3.2.6. Therefore we have by the conjugacy (3.3) that $\widetilde{u}(*_1+\epsilon) \in C(u(LI^\alpha(\alpha)|_M))$, and $\lim_{\epsilon\to 0} \widetilde{u}(*_1 + \epsilon) = \widetilde{u}*_1 = y$. Here we have that $\widetilde{u}$ is continuous at $*_1$ because $*_1$ is not in the postcritical set P_α (see (3.2) and the surrounding discussion). This proves the claim, and completes the proof of (3.7).

Now we turn to the proof of the last statement. In the last item above we showed that $\widetilde{u}*_1, \widetilde{u}*_2 \in \overline{C(g^L|_M)}$ for all M. Similar arguments show that $\widetilde{u}*_1, \widetilde{u}*_2 \in \overline{C(g^R|_M)}$ for all M, and this proves that

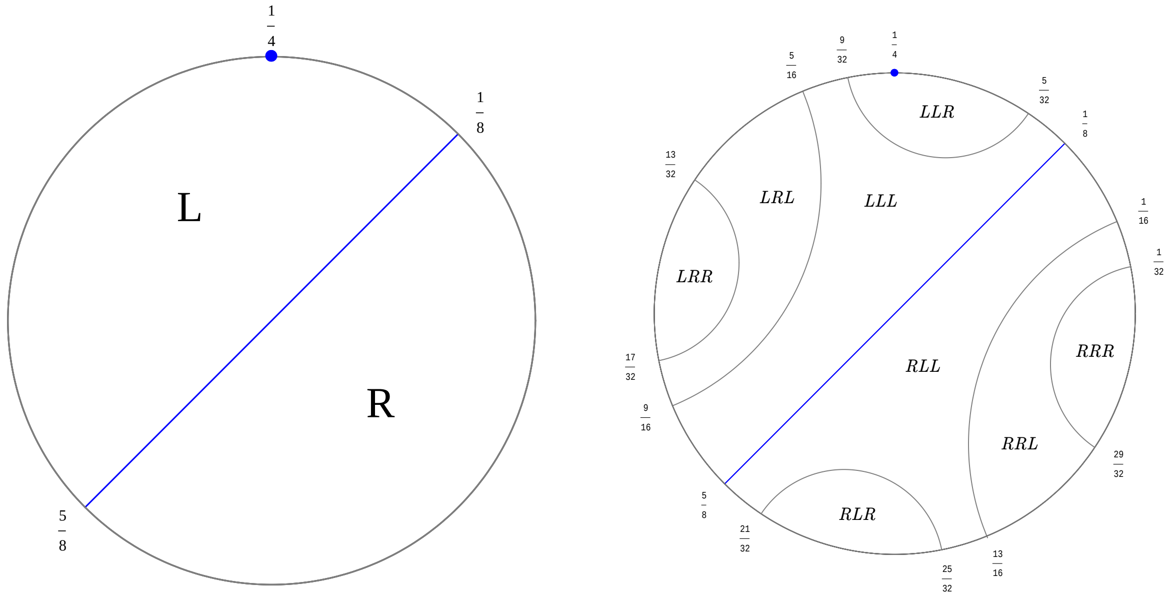
$$\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)} \supset \widetilde{u}\{*_1, *_2\}.$$

For the reverse inclusion, suppose $y \in \bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)}$. Then $y \in \overline{C(g^L|_{|u|+1})} = \overline{C(uL)}$, therefore $h^{|u|}y \in \overline{h^{|u|}C(uL)} = \overline{C(L)} = \overline{L}$ by continuity and closedness of h. Similarly $h^{|u|}y \in \overline{R}$. This shows that $h^{|u|}y \in \overline{L} \cap \overline{R} = \{*_1, *_2\}$.

On the other hand $y \in \overline{C(g^L|_{|u|})} = \overline{C(u)}$, so $y \in \widetilde{u}\{*_1, *_2\}$ as desired. $\square$

By (3.3), the cylinder $C(g)$ can also be described as the image of $\mathbb{T}\backslash A_g$ under $\widetilde{g}$ (recall the definition of A_g in (3.2)), that is $C(g) = \widetilde{g}(\mathbb{T}\backslash A_g)$. Since $\widetilde{g}$ is continuous on each component of $\mathbb{T}\backslash A_g$, this shows that $C(g)$ is a finite union of disjoint open intervals. Induction on the length of g shows that the closure of these intervals is disjoint too (nonperiodicity of α is needed here).

By keeping track of when a cylinder contains α, one sees that boundary chords of cylinders are always leaves in the minimal equivalence (see Proposition 3.2.2).



(a) The left and right halves L and R of S^1. Here $\alpha = 1/4$ and so $I^\alpha(\alpha) = LLRRR\ldots$.

(b) The collection of cylinders $C(g)$ for $g \in \{L,R\}^3$ partitions $\mathbb{T}$ up to a finite set. The boundary leaves of $C(LLL)$ are $\{*_1 = 1/8, *_2 = 5/8\}$, $\widetilde{L}\{*_1,*_2\} = \{5/16, 9/16\}$ and $\widetilde{LL}\{*_1,*_2\} = \{5/32, 9/32\}$, as we expect from Proposition 3.2.12.

Proposition 3.2.12. *Suppose $g \in \{L,R\}^N$ is a finite word of length N. Then the boundary chords of $C(g)$ are all of the form $\widetilde{g|_t}\{*_1,*_2\}$ where $0 \le t \le N-1$. Moreover, $\widetilde{g|_t}\{*_1,*_2\}$ is a boundary chord iff $\sigma^{t+1}g$ is an initial subword of $I^\alpha(\alpha)$.*

Proof. We proceed by induction on $N = |g|$. For $N = 1$, the result is clear because $\{*_1,*_2\}$ is the boundary chord of $C(L)$ and of $C(R)$. Now suppose $sg \in \{L,R\}^{N+1}$ where $s \in \{L,R\}$ and $g \in \{L,R\}^N$. By the induction hypothesis, the boundary chords of $C(g)$ are precisely the leaves of the form $\widetilde{g|_t}\{*_1,*_2\}$ where $0 \le t \le N-1$ is an integer such that $\sigma^{t+1}g$ is an initial subword of $I^\alpha(\alpha)$.

The images of the boundary chords of $C(g)$ under $\widetilde{s}$ are always boundary chords of

$\widetilde{s}(C(g)) = C(sg)$. By the induction hypothesis, all leaves arising in this way are of the form $\widetilde{s} \circ \widetilde{g|_t}\{*_1, *_2\}$ where $0 \leq t \leq N-1$ is an integer such that $\sigma^{t+1} g = I^\alpha(\alpha)|_{N-t-1}$. Note that $1 \leq t+1 \leq N$, that $\widetilde{s} \circ \widetilde{g|_t} = \widetilde{(sg)|_{t+1}}$, and $\sigma^{(t+1)+1} sg$ is an initial subword of $I^\alpha(\alpha)$.

If $\alpha \notin C(g)$ then these are the only boundary chords of $C(sg)$. On the other hand if $\alpha \in C(g)$ then $\{*_1, *_2\}$ is a new boundary chord of $C(sg) = \widetilde{s}(C(g))$. This new boundary chord is equal to $\{*_1, *_2\} = \widetilde{sg|_0}\{*_1, *_2\}$, and $\sigma^{0+1}(sg) = g$ is an initial subword of $I^\alpha(\alpha)$ because $\alpha \in C(g)$ means that $I^\alpha(\alpha)|_n = g$.

This completes the induction. $\square$

As promised, combining the previous two propositions shows that every equivalence class in $\approx_\alpha$ can be approximated by boundary leaves.

Proposition 3.2.13. *Let $g \in \{L, R\}^\infty$ be an infinite word. Then $C(g) := \bigcap_n \overline{C(g|_n)} = \{x_1, \ldots, x_m\}$ is finite. Assume the $x_1, \ldots, x_m$ are arranged in a counterclockwise order. Then for each i, and each $\epsilon > 0$ there is an integer n such that $\widetilde{g|_n}\{*_1, *_2\}$ is ϵ-close to $\{x_i, x_{i+1}\}$ in the Hausdorff sense.*

Proof. By Proposition 3.2.11, all the points of $\bigcap_n \overline{C(g|_n)}$ belong to the same $\approx_\alpha$ class, so by Theorem 3.2.3, $\bigcap_n \overline{C(g|_n)}$ is finite, and we can write $\bigcap_n \overline{C(g|_n)} = \{x_1, \ldots, x_m\}$.

Let $U = \cup_i (x_i - \epsilon, x_i + \epsilon)$ be the union of ϵ-balls around each point in $C(g)$. Assume ϵ is small enough that these balls are disjoint. By Lemmas 3.2.10 and 3.2.9, we have for sufficiently large M that $U \supset \overline{B_M(x)} \supset \overline{C(g|_M)} \supset \{x_1, \ldots, x_m\}$.

Fix M sufficiently large as above, and let (a, b) be a component of $\mathbb{T} \backslash \{x_1, \ldots, x_m\}$. Let z be a point of $\mathbb{T} \backslash U$ inside (a, b), and let I be the component of $\mathbb{T} \backslash \overline{C(g|_M)}$ containing z. By construction, the endpoints of I are within distance ϵ of x_i and x_{i+1}, and by Proposition 3.2.12, the endpoints of I are of the form $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ where u is an initial subword of $g|_M$. $\square$

In fact, in Lemma 3.3.2, we will need to construct several *distinct* approximations to the equivalence class $[\approx_\alpha]$, so it is important our approximations can be chosen to be strict approximations.

Lemma 3.2.14. *Let $g = LI_\alpha(\alpha)$ or $g = RI_\alpha(\alpha)$ so that $C(g) = [*_1] \cap \overline{L}$ or $C(g) = [*_2] \cap \overline{R}$ respectively.*

Let x_i, x_{i+1} be adjacent elements of $C(g)$. If either of the following conditions hold, the approximations in the conclusion of Proposition 3.2.13 may be taken to be strict in the sense that $\widetilde{g|_n} *_1$ and $\widetilde{g|_n} *_2$ are not equal to any of the elements in $C(g)$.

- *At least one of the x_i, x_{i+1} is not in $\{*_1, *_2\}$.*
- *$\{x_i, x_{i+1}\} = \{*_1, *_2\}$, and $[*_1] = \{*_1, *_2\}$.*

In the case not covered by this lemma, where $\{x_i, x_{i+1}\} = \{*_1, *_2\}$ and $|[*_1]| > 2$, it is impossible to find strict approximations, as this would violate flatness of $\approx_\alpha$.

Proof. Let $x_i, x_{i+1} \in C(g)$ be the chord that we wish to approximate.

Suppose first that one of the x_i, x_{i+1} is not in $\{*_1, *_2\}$. Let $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ be the approximating leaf of $\{x_i, x_{i+1}\}$ guaranteed by Lemma 3.2.13. Assume for contradiction that $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ intersects $[*_1]$. Then $\widetilde{u}*_1 \sim_\alpha *_1$. Applying $h^{|u|+1}$ to both sides yields $[\alpha] = h^{|u|}[\alpha]$, so since $[\alpha]$ is nonperiodic we must have $|u| = 0$. If ϵ is sufficiently small, this contradicts the fact that $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ approximates $\{x_i, x_{i+1}\}$, because $\{\widetilde{u}*_1, \widetilde{u}*_2\} = \{*_1, *_2\}$ is not equal to $\{x_i, x_{i+1}\}$, since we assumed that $x_{i+1} \notin \{*_1, *_2\}$.

Now we consider the case where x_i and x_{i+1} are both in $\{*_1, *_2\}$, and $[*_1] = \{*_1, *_2\}$. Assume first that $g = LI^\alpha(\alpha)$, the other case is similar.

We will go through the construction of Proposition 2.4.4 again, making a small modification. Fix M large enough that $\overline{C(g|_M)}$ is contained in an ϵ-neighborhood of $\{*_1, *_2\}$. Assume that ϵ is small enough that this neighborhood does not contain α.

Let I be the component of $\mathbb{T} \backslash \overline{C(g|_M)}$ that contains α. Then I is contained inside $L = (*_1, *_2)$.

Consider the boundary chord l joining the endpoints of I, which by Proposition 3.2.12 is of the form $\widetilde{u}\{*_1, *_2\}$ where u is an initial subword of $g|_M$. Note that I is not equal to L because $C(g|_M) \cap L = C(g|_M)$ has positive total length, so $\widetilde{u}$ is not the identity and u is not the empty word. In particular, the boundary chord $l = \widetilde{u}\{*_1, *_2\}$ is a strict approximation (close, but not equal) of $\{*_1, *_2\}$. □

3.3 All scales are good

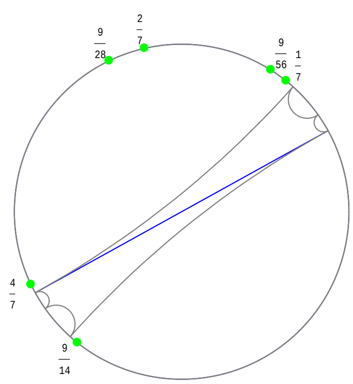
In this section we prove Theorem 3.1.5, that is for α combinatorially semihyperbolic, we show that the equivalence relation $\sim_\alpha$ is qs-good at all scales. It suffices to consider dyadic scales $r = 2^{-N}$.

The construction is sketched in Figure 3.2 and 3.3. The idea is to first show that $\sim$ is qs-thick across intervals around $*_1$ and $*_2$ at scale $N = 1$, this is Proposition 3.3.3. To get the (uniformly perfect) Cantor set A for the gluing between $*_1$ and $*_2$, we will use periodic leaves near $*_1$ and $*_2$ to generate a linear iterated function system. The existence of such periodic leaves is shown in Lemma 3.3.2. See Figure 3.2a. After this gluing at the large scale is constructed, we use backwards iteration to get the Cantor set around any point at any scale. This existence of backwards iterates that transport the gluing at large scale to a gluing at an arbitrary point $x \in \mathbb{T}$ and arbitrary scale $r = 2^{-N}$ is given by the construction of circular chains, see Definition 3.3.5 and Lemma 3.3.7.

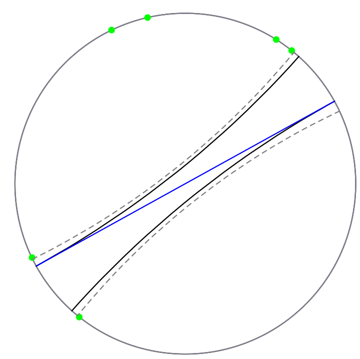
The construction will rely on the following fact that the class of the critical point $[*_1]$ is contained in the union of exactly two components I' and I'' of $\mathbb{T}\backslash\overline{P_\alpha}$, where we recall that $P_\alpha = \{h^t\alpha : t \geq 0\}$. In particular this means that all compositions $\widetilde{g}$ of $\widetilde{L}$s and $\widetilde{R}$s are well defined on I' and I'' (see (3.2)), and hence the images $\widetilde{g}(I')$ and $\widetilde{g}(I'')$ are connected intervals. See Figure 3.2a. This guarantees that if we can construct the good gluing across the neighborhoods I',I'', then we can pull them back to different scales and locations via the inverse branches of h^n.

Proposition 3.3.1. *Let I' and I'' be the components of $\mathbb{T}' := \mathbb{T}\backslash\overline{P_\alpha}$ containing $*_1$ and $*_2$ respectively. Then I' and I'' are distinct, and $[*_1] \subset I' \cup I''$.*

Note that the combinatorial semihyperbolicity (Definition 3.2.5) assumption on α, together with Lemma 3.2.6, implies that $*_1, *_2 \notin \overline{P_\alpha}$. To see this, suppose for contradiction

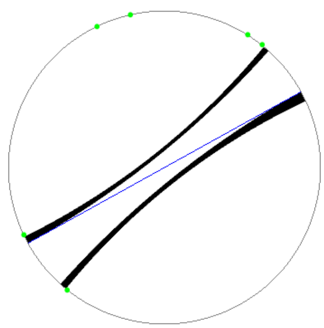


(a) The postcritical set P_α is marked (green) on $\mathbb{T}$. The diameter $D = \{*_1, *_2\}$ is in blue. The hyperbolic geodesics denote leaves in $\sim_\alpha$. Here $\alpha = 9/56$, $I' = [9/14, 1/7]$ and $I'' = [4/7, 9/14]$. Notice that $[*_1] \subset I' \cup I''$.

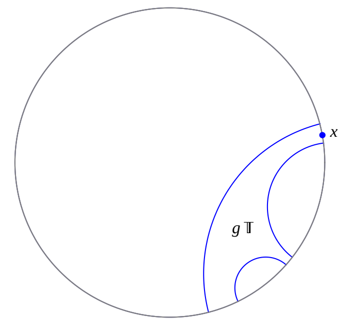


(b) Periodic leaves (dashed) with endpoints in I' and I'' are constructed (Lemma 3.3.2). Here we only show two leaves; the actual construction requires two pairs of such leaves.

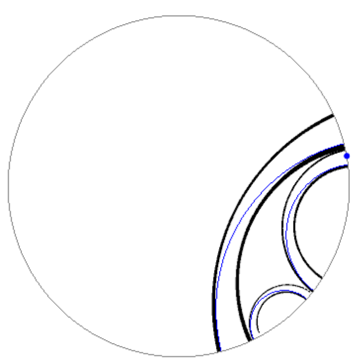
Figure 3.2: The steps involved in the proof of Theorem 3.1.5, in the case $\alpha = 9/56$. Continued in Figure 3.3.



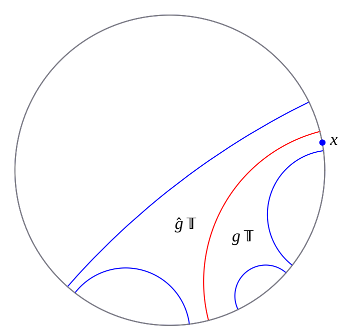
(a) We use the periodic leaves near to the diameter to construct an IFS, resulting in a gluing between the intervals I_1 and I_4 and between I_2 and I_3. (Proposition 3.3.3). The gluing is resprented by the thick black family of geodesics, but in actuality the gluing is only supported on a linear Cantor set.



(b) We construct the gluing at $x \in \mathbb{T}$ and scale 2^{-N} by pulling back the gluing from the previous step. The cylinder of depth N containing x is bounded by leaves of the form $\widetilde{g|_{t_j}}\{*_1, *_2\}$ for some integers $t_j \leq N$, where $g = I^\alpha(x)$...



(c) ...so, as long as t_j is not too small, $\widetilde{g|_{t_j}} I_1$ and $\widetilde{g|_{t_j}} I_4$ (and $\widetilde{g|_{t_j}} I_2$ and $\widetilde{g|_{t_j}} I_3$) have diameter comparable to 2^{-N}. Note that even though the figure shows gluings inside the cylinder, we do not use them in this construction - we only use the gluings on the exterior of the cylinder.



(d) Sometimes one of the boundary leaves (the red one) is of the form $g^{N-t}\{*_1, *_2\}$ where t is too large, so that pulling back the gluing from I_1 to I_4 and I_2 and I_3 results in a gluing at too large a scale. In this case we use the leaves of the gap $\hat{g}\mathbb{T}$ instead of the red leaf.

Figure 3.3: The steps involved in the proof of Theorem 3.1.5, in the case $\alpha = 9/56$. Continued from Figure 3.2.

that $h^{t_n}(\alpha) \to *_i$ for some sequence $t_n \to \infty$. From Lemma 3.2.6 this implies for sufficiently large n that $\sigma^{t_n} I^\alpha(\alpha) = I^\alpha(h^{t_n}\alpha) \overset{M_\alpha+1}{\asymp} I^\alpha(*_i) = \star I^\alpha(\alpha)$, where M_α is the combinatorial semihyperbolicity constant. Therefore $\sigma^{t_n+1} I^\alpha(\alpha) \overset{M_\alpha}{\asymp} I^\alpha(\alpha)$ and this contradicts combinatorial semihyperbolicity.

In particular, the components I' and I'' do exist.

Proof. Since the kneading sequence $I^\alpha(\alpha)$ is not periodic, it is not $LLL\ldots$. Therefore there exists a postcritical point $h^t\alpha \in R$. Then α and $h^t\alpha$ are in different components of $\mathbb{T}\backslash\{*_1, *_2\}$, which means that $*_1$ and $*_2$ are in different components of $\mathbb{T}\backslash\{\alpha, h^t\alpha\}$, which means $*_1$ and $*_2$ are in different components of $\mathbb{T}\backslash P_\alpha$. Thus I' and I'' are distinct.

Now we turn to the second statement, which is equivalent to saying that all the postcritical points P_α lie in two components of $\mathbb{T}\backslash[*_1]$.

The idea is that if a postcritical point $h^t *_1$ lies in a given component of $\mathbb{T}\backslash[*_1]$, then by flatness the whole class $h^t[*_1]$ of that point must lie in that same component. But since h is expanding on $\mathbb{T}$, any component of $\mathbb{T}\backslash[*_1]$ that contains an iterate $h^t[*_1]$ of the critical class must necessarily be 'large'. The desired result follows if we can show that only two components are 'large'.

More precisely, we will now show that if J is a component of $\mathbb{T}\backslash[*_1]$ and J contains a postcritical point $h^t\alpha$, $t \geq 1$, then the length of J satisfies $|J| > 1/4$.

For $x, y \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ let $|x - y| \in [0, 1/2]$ be the distance between x and y on $\mathbb{T}$, which is the (normalized) length of the shortest arc joining x to y. Then the distance between any pair of points $x, y \in \mathbb{T}$ changes under the action of h according to the tent mapping:

$$|hx - hy| = \begin{cases} 2|x-y| & \text{if } |x-y| \leq 1/4 \\ 1 - 2|x-y| & \text{if } |x-y| > 1/4. \end{cases} \tag{3.8}$$

Let $*'$ and $*''$ be adjacent points of $[*_1]$, bounding some component J of $\mathbb{T}\backslash[*_1]$. Let $d = |*' - *''| = |J|$, and suppose $d \leq 1/4$. Then we have $|h*' - h*''| = 2d$. By (3.8), iterating h on the leaf $\{*', *''\}$ will yield longer and longer leaves until the length is greater than $1/4$.

(The length of a leaf is defined to be the distance between its endpoints). After that point the length of the leaf may shrink.

However, the longest leaf in the lamination has length at most $1/2 - d$. This is because, by flatness, any leaf in the lamination must have both endpoints in the same component of $\mathbb{T}\backslash[*_1]$. The points $*' \approx_\alpha *'' \approx_\alpha *' + 1/2 \approx_\alpha *'' + 1/2$ are all in $[*_1]$ (the easiest way to see this is by considering itineraries), and the largest component of $\mathbb{T}\backslash\{*', *'', *' + 1/2, *'' + 1/2\}$ has length $1/2 - d$, so the largest component of $\mathbb{T}\backslash[*_1]$ has length at most $1/2 - d$.

So from (3.8), the iterates $h^t\{*', *''\}$ of the leaf never get shorter than $2d = 2|J|$. In particular, for $t \geq 1$, h^t*' and h^t*'' can never both be in J. Thus, by flatness, $h^t[*'] = h^t[*_1]$ is never contained in J.

We have shown that if a component of $\mathbb{T}\backslash[*_1]$ is shorter than $1/4$, then it contains no postcritical points. Now let us use this fact to derive the desired result. Let J be the largest component of $\mathbb{T}\backslash[*_1]$. Then $|J| > 1/4$. Since $[*_1] = [*_1]_{\approx_\alpha}$ is invariant under $x \mapsto x + 1/2$ (again, by considering itineraries), the interval $J + 1/2$ is a component of $\mathbb{T}\backslash[*_1]$ and also has length equal to $|J| > 1/4$. This shows that components of length greater than $1/4$ occur in pairs. Since $(|J| + |J + 1/2|) + (1/4 + 1/4) > 1$, there can only be one pair of components of $\mathbb{T}\backslash[*_1]$ with length greater $1/4$, namely J and $J + 1/2$. Therefore the postcritical points all lie in the two components J and $(J + 1/2)$, and the result follows. $\square$

A *periodic leaf* is a leaf such that both endpoints have periodic itineraries. We now show that we can find a pair of periodic leaves spanning the intervals I' and I''. These leaves will be used to generate an iterated function system, giving many leaves between I' and I''.

Lemma 3.3.2 (Existence of periodic leaves near the main leaf $(*_1, *_2)$)**.** *There exist infinitely many distinct periodic leaves l' with both endpoints in L with one endpoint in I' and the other endpoint in I''. The same statement holds for R in place of L.*

Proof. We will only construct such leaves in L, the argument for getting leaves in R is exactly the same. First we prove that we can find a leaf l in L with one endpoint in I' and the other endpoint in I''. This leaf will be of the form $\widetilde{u}\{*_1, *_2\}$ for some $u \in \{L, R\}^*$. If u is sufficiently contracting then this tells us that $\widetilde{u}^2$ maps I' into I' and I'' into I''. We then use the contraction principle to find fixed points a', a'' of $\widetilde{u}^2$ in I' and I'' respectively. By definition, a' and a'' will be periodic with periodic itinerary equal to u^∞. In particular $a' \approx_\alpha a''$. Now we provide the details.

1. Let $g = I^\alpha(*_1) = \star I^\alpha(\alpha)$. By Proposition 3.2.11, we have

$$[*_1] = \bigcap_n \overline{C(g^L|_n)} \cup \bigcap_n \overline{C(g^R|_n)} = \bigcap_n \overline{C(LI^\alpha(\alpha))} \cup \bigcap_n \overline{C(RI^\alpha(\alpha))}.$$

   Since, by definition, $C(L\cdots)$ and $C(R\cdots)$ are contained in L and R respectively, we conclude that $[*_1] \cap L \subset \bigcap_n \overline{C(g^L|_n)}$. Taking the closure of both sides yields $[*_1] \cap \overline{L} \subset \bigcap_n \overline{C(g^L|_n)}$. But the reverse inclusion holds too, so we get

$$[*_1] \cap \overline{L} = \bigcap_n \overline{C(g^L|_n)}. \tag{3.9}$$

   Let $[*_1] \cap \overline{L} = \{y_1, \ldots, y_m\}$ where the y_i are assumed to be indexed in counterclockwise order with $y_1 = *_1$ and $y_m = *_2$. Let i be the maximal index such that $y_i \in I'$. Then $y_{i+1} \in I''$ by Proposition 3.3.1.

   By Theorem 3.2.13, the leaf $\{y_i, y_{i+1}\}$ is the limit of leaves of the form $\{\widetilde{g_n} *_1, \widetilde{g_n} *_2\}$ for a sequence of finite words $g_n \in \{L, R\}^*$. All these words are initial subwords of the word $g^L = LI^\alpha(\alpha)$, and by Lemma 3.2.14 we can assume that $\lim |g_n| = \infty$.

2. Since $\{\widetilde{g_n} *_1, \widetilde{g_n} *_2\}$ converges to a leaf with endpoints in I' and I'', we have for sufficiently large n that $\widetilde{g_n}$ maps $*_1$ into I' and $*_2$ into I'', or $*_1$ into I'' and $*_2$ into I'. By replacing g_n with $g_n g_n$ we can assume that the former occurs.

   Since $\lim_n |g_n| = \infty$, we have $\lim_n 2^{-|g_n|} = \lim_n |g_n'| = 0$, so for sufficiently large n we have that $\widetilde{g_n}$ maps I' into I' and I'' into I''.

   Also, $\widetilde{g_n}$ has constant derivative $2^{-|g_n|} < 1$, so by the contraction principle $\widetilde{g_n}|_{I'}$ and $\widetilde{g_n}|_{I''}$ have fixed points a', a'' in I', I'' respectively. Using (3.3) we see that

$$I^\alpha(a') = I^\alpha(\widetilde{g_n} a') = g_n I^\alpha(a'),$$

and hence a' has periodic itinerary $I^\alpha(a') = I^\alpha(a'') = g_n^\infty \in \{L, R\}^\infty$. The same argument shows that $I^\alpha(a'') = g_n^\infty$ too. Thus $\{a', a''\}$ is a periodic leaf with one endpoint in I' and the other endpoint in I'', and $\{a', a''\} \subset L$.

Step 2) shows that every sufficiently large n yields a periodic leaf with endpoints in I' and I''. For our construction below we will need two different periodic leaves, to form an iterated function system. We will now show that we can choose $n' \neq n$ such that the applying the construction in Step 2) to $g_{n'}$ and g_n yields different periodic leaves. We do this by choosing n' and n so that $g_n^\infty \neq g_{n'}^\infty$. Then the periodic leaves that result from applying step 3) to g_n and $g_{n'}$ will be different because they will have different itineraries and hence will not intersect.

We need to consider two cases.

Suppose first that $g := LI^\alpha(\alpha)$ is eventually periodic (recall that g cannot be actually periodic), which means there exists some preperiod $t \geq 1$ such that $I^\alpha(h^t\alpha)$ is periodic of some period $K \geq 1$. Choose M large enough that M is greater than the preperiod t of $I^\alpha(\alpha)$, and large enough that the first M letters of $LI^\alpha(\alpha)$ are enough to 'certify' that $g = LI^\alpha(\alpha)$ is not periodic of period K. That is, choose $M \geq t$ large enough that

$$LI^\alpha(\alpha)|_M \text{ is not of the form } w^\infty|_M \text{ for any } w \in \{L, R, \star\}^K.$$

Let n be large enough that $|g_n| \geq M$, and let n' be large enough that $|g_{n'}| - |g_n| \geq M$. Write $g'_n = g_n u$ where $|u| = |g_{n'}| - |g_n| \geq M$ (recall that g_n and $g_{n'}$ are both initial subwords of $g = LI^\alpha(\alpha)$). Suppose for contradiction that $g_n^\infty = g_{n'}^\infty$, then applying the shift $\sigma^{|g_n|}$ to both sides yields $g_n^\infty = u g_{n'}^\infty$.

Since g_n is an initial subword of g of length at least M, and u has length at least M, the equality $g_n^\infty = u g_{n'}^\infty$ implies $g|_M = u|_M$. However, u is an initial subword of something that is periodic of period K (namely, $\sigma^{|g_n|}g$), so this contradicts the definition of M. Therefore $g_n^\infty \neq g_{n'}^\infty$.

Now we consider the case where $g = LI^\alpha(\alpha)$ is not eventually periodic. Fix n, and choose M large enough that the first M letters of $\sigma^{|g_n|}g$ are enough to 'certify' that $\sigma^{|g_n|}g$ is not periodic of period $|g_n|$. That is, choose M large enough that

$$(\sigma^{|g_n|}g)|_M \text{ is not of the form } w^\infty|_M \text{ for any } w \in \{L, R, \star\}^{|g_n|}.$$

Then choose n' large enough that $|g_{n'}| - |g_n| \geq M$. Write $g'_n = g_n u$ where $|u| = |g_{n'}| - |g_n| \geq M$ (recall that g_n and $g_{n'}$ are both initial subwords of $g = LI^\alpha(\alpha)$). Suppose for contradiction that $g_n^\infty = g_{n'}^\infty$, then applying the shift $\sigma^{|g_n|}$ to both sides yields $g_n^\infty = u g_{n'}^\infty$.

Restricting to the first M letters yields $g_n^\infty|_M = u|_M$. Now, u is an initial subword of $\sigma^{|g_n|}g$, so we get $g_n^\infty = (\sigma^{|g_n|})g|_M$, which contradicts the definition of M. $\square$

The points $*_1$ and $*_2$ cut I' and I'' respectively into two open subintervals each, giving a total of four intervals $J_1^-, J_2^-, J_2^+, J_1^+$ (assumed to be in counterclockwise order). Here we choose the indexing on the $\{J_j\}$ such that $*_1$ is the counterclockwise endpoint J_1^- and the clockwise endpoint of J_2^-, and $*_2$ is the counterclockwise endpoint of J_2^+ and the clockwise endpoint of J_1^+. Thus $J_1^- \cup J_2^- = I'$ and $J_1^+ \cup J_2^+ = I''$, and $J_1^- \cup J_1^+ \subset R$ while $J_2^- \cup J_2^+ \subset L$.

Let l be a periodic leaf in $\sim_\alpha$ of period p, where the period is defined as the smallest integer p such that h^p fixes both endpoints of l. Observe that the iterates of each endpoint never lie in $[*_1]$, as this would contradict nonperiodicity of $[*_1]$. In other words neither itinerary is precritical, therefore by (3.4) , the itineraries of both points are equal, and we will use $I^\alpha(l)$ to denote this common itinerary.

The common itinerary of both points is periodic of period p, so we can write $I^\alpha(l) = w^\infty$ where $w = I^\alpha(l)|_p$. Observe that $\widetilde{w}$ is a contraction that fixes the endpoints of l, indeed $\widetilde{w}$ is just the inverse branch of h^p that fixes the endpoints of l. If we let $w = I^\alpha(l)|_{2p}$ then $\widetilde{w}$ is orientation preserving and still fixes the endpoints of l, and we will assume this.

The contractions arising from the periodic leaves we constructed in Lemma 3.3.2 generate an iterated function system, giving us a Cantor set on which a gluing between I' and I'' is supported (Recall the definitions of Section 2.5).

Proposition 3.3.3 (Existence of Cantor set around main leaf)**.** *J_1^- is qs-glued to J_1^+ and J_2^- is qs-glued to J_2^+. In other words, $r = 1$ is a qs-good scale of degree 2 at $*_1$ (and $*_2$).*

Proof. First we show that J_1^- is glued to J_1^+. From Lemma 3.3.2 we get a pair of distinct periodic leaves l, l' each with one endpoint in J_1^- and the other endpoint in J_1^+. Let a^-, a^+ be the endpoints of l in J_1^- and J_1^+ respectively, and let a'^-, a'^+ be the endpoints of l' in J_1^- and J_1^+ respectively. Let H_1^- be the closed subinterval of J_1^- with endpoints $\{a^-, a'^-\}$ and H_1^+ be the closed subinterval of J_1^+ with endpoints $\{a'^+, a^+\}$. Let $\varphi : H_1^- \to H_1^+$ be the linear (w.r.t. arclength) map that takes a^- to a^+ and a'^- to a'^+.

As in the above discussion, let $w = I^\alpha(l)|_{2p}$ and $w' = I^\alpha(l')|_{2p'}$ where p and p' are the periods of l and l' respectively. Recall that $\widetilde{w}$ fixes a^- and a^+ while $\widetilde{w'}$ fixes a'^- and a'^+, and $\widetilde{w}$ and $\widetilde{w'}$ are orientation preserving.

Consider the restrictions $\widetilde{w}|_{H_1^-}$ and $\widetilde{w}|_{H_1^+}$ of $\widetilde{w}$ to H_1^- and H_1^+ respectively. They are conjugate:

$$\widetilde{w}|_{H_1^+} \circ \varphi = \varphi \circ \widetilde{w}|_{H_1^-}. \tag{3.10}$$

To see this, note that both sides of the equation are linear mappings with the same derivative, and they are equal when evaluated at a^-, and they are both of the same orientation class.

Similarly, $\widetilde{w'}|_{H_1^+} \circ \varphi = \varphi \circ \widetilde{w'}|_{H_1^-}$ holds for the linear maps induced by the other periodic leaf, l'.

Consider the linear iterated function systems generated by the contractions $\widetilde{w}$ and $\widetilde{w'}$ on each of the intervals H_1^- and H_1^+. Let A_1^- and A_1^+ be the respective limit sets. That is, A_1^- is the closure of the orbit of the two endpoints $\partial H_1^- = \{a^-, a'^-\}$ of H_1^- under arbitrary finite compositions of $\widetilde{w}|_{H_1^-}$ and $\widetilde{w'}|_{H_1^-}$, and similarly for A_1^+. Since $\widetilde{w}, \widetilde{w'}$ are linear contractions on an interval, the limit set is a Cantor set and it is an easy exercise to show that the limit sets are uniformly perfect.

We claim that $x \sim \varphi(x)$ for all $x \in A_1^-$. This is true by definition when $x = a^-$ or $x = a'^-$. On the other hand, backwards invariance of $\sim$ and the conjugacy (3.10) yields, for $x \in H_1^-$:

$$x \sim \varphi(x) \implies \{ \quad \widetilde{w}x \sim \widetilde{w}\varphi(x) = \varphi(\widetilde{w}x) \quad \text{and} \quad \widetilde{w'}x \sim \widetilde{w'}\varphi(x) = \varphi(\widetilde{w'}x) \quad \}.$$

By induction, we get $x \sim \varphi(x)$ for all images of a^- and a^+ under arbitrary finite compositions of $\widetilde{w}$ and $\widetilde{w'}$. By topological closedness of $\sim$ we get $x \sim \varphi(x)$ for all $x \in A_1^-$.

Thus we have constructed a gluing between J_1^- and J_1^+ where φ is not only quasisymmetric but linear, and A_1^-, A_1^+ are not only uniformly perfect, but are linear Cantor sets. A similar argument shows that J_2^- and J_2^+ are glued together on some Cantor sets A_2^- and A_2^+. Thus, $r = 1$ is a good scale at $*_1$ (and $*_2$), $\square$

To get the gluing at scale $r = 2^{-N}$ and $x \in \mathbb{T}$, we will pullback the gluing we just constructed. For certain x at certain scales, this can be done fairly directly.

Lemma 3.3.4. *Fix $N > 0$ integer. Suppose $x \in \mathbb{T}$ and let $g = I^\alpha(x)$. If $g|_N = u \star I^\alpha(\alpha)|_N$ where $N - M_\alpha \leq |u| \leq N$, then $r = 2^{-N}$ is a L-qs-good scale for x, where L only depends on M_α (the combinatorial semihyperbolicity parameter)*

Proof. We see that $\widetilde{u}$ maps $*_1$ or $*_2$ to x. By backwards invariance of $\sim$ and linearity of $\widetilde{u}$, we can use $\widetilde{u}$ to pullback the gluing chain constructed in Proposition 3.3.3 to a chain of degree two around x. Since $N - M_\alpha \leq |u| \leq N$, the backwards iterate $\widetilde{u}$ maps I' and I'' onto intervals of size comparable to 2^{-N}. See the proof of Theorem 3.1.5 below for details. $\square$

For general x and N, the construction is more complicated. For instance we will need to use gluing chains of degree higher than 2. For this purpose it is useful to introduce the notion of a *circular chain.*

Definition 3.3.5 (Circular chain)**.** For $m \geq 2$, if $l_1, \ldots, l_m$ are mutually non-intersecting chords in the lamination $\sim$, let $\mathrm{Gap}(l)$ be the component of $\overline{\mathbb{D}} \setminus \cup_i l_i$ that contains the convex hull of the $2m$ endpoints of the l_i. We say that $l_1, \ldots, l_m$ form an ϵ-*circular chain* around $x \in \mathbb{T}$ if

- All the chords $l_1, \ldots, l_m$ lie on the boundary of $\mathrm{Gap}(l)$.
- All the components of $\mathrm{Gap}(l) \cap \mathbb{T}$ have length bounded above by ϵ.
- The *interior* of the circular chain, $\mathrm{Gap}(l) \cap \mathbb{T}$, contains x.

For $m = 1$, we say that l_1 forms a ϵ-circular chain around x if both endpoints of l_1 are within ϵ of x, and if x is contained in the interior of the circular chain. In this case ($m = 1$), the *interior* is the smallest component of $\overline{\mathbb{D}} \backslash \{l_1\}$.

Recall that M_α is the quantitative parameter in the definition of combinatorial semihyperbolicity.

For each $x \in \mathbb{T}$ and each scale $r = 2^{-N}$, we will construct a $\asymp 2^{-N}$-circular chain around x by using boundary leaves of cylinder sets. By Proposition 3.2.12, these boundary leaves are all pullbacks of the main leaf $(*_1, *_2)$ under the doubling map h.

Thus for each leaf l_i of the circular chain, we can pullback the gluing around the main leaf (Proposition 3.3.3) to a gluing around l_i.

To make this work we need to ensure that we can construct the desired circular chain with a bounded number of boundary leaves.

Lemma 3.3.6 (Bound on the number of boundary leaves)**.** *Suppose $g \in \{L, R\}^N$ is a finite word of length N. There can be at most one integer $t < N - M_\alpha$ such that $\widetilde{g|_t}\{*_1, *_2\}$ is a boundary leaf of $C(g)$. In particular the number of boundary leaves of $C(g)$ is bounded by $M_\alpha + 1$.*

Proof. Suppose for contradiction $t, t' \leq N - M_\alpha - 1$ are distinct integers such that $\widetilde{g|_t}\{*_1, *_2\}$ and $\widetilde{g|_{t'}}\{*_1, *_2\}$ are boundary leaves of $C(g)$, and assume without loss of generality that $t < t'$. By Proposition 3.2.12, we have $g = g|_t s v = g|_{t'} s' v'$ where $s, s' \in \{L, R, \star\}$ and v, v' are initial subwords of $I^\alpha(\alpha)$. Applying the shift σ^{t+1} yields $v = (\sigma^{t+1} g|_{t'}) s' v'$. This shows that $\sigma^T v = v'$ where $T > 0$. Since $t' \leq N - M_\alpha - 1$, we have that $|v'| \geq M_\alpha$, so the last equality contradicts combinatorial semihyperbolicity. ∎

With this lemma, we can now prove the existence of circular chains around x.

Lemma 3.3.7 (Existence of chains around x)**.** *There exists $C > 0$ such that the following holds. For each $x \in \mathbb{T}$ and each $N \geq 0$, either the hypotheses of Lemma 3.3.4 are satisfied, or, there exists m finite words $u(1), \ldots, u(m) \in \{L, R\}^*$ such that*

- $m \leq 2M_\alpha$
- *The lengths of the words satisfy* $N - M_\alpha \leq |u(i)| \leq N$
- *The leaves* $l_i := \widetilde{u(i)}\{*_1, *_2\}$ *form a* $C2^{-N}$*-circular chain around* x *in the sense of Definition* 3.3.5.

Proof. Suppose $x \in \mathbb{T}$ and $N \geq 0$. Let $g = I^\alpha(x)$ be the itinerary of x. There are four cases to consider.

1. $g = u \star I^\alpha(\alpha)$ for some finite word $u \in \{L, R\}^*$, and $|u| \geq N - M_\alpha$.
2. The symbol $\star$ does not occur in g, and if u is a word such that $|u| \leq N - 1$ and $g|_N = usI^\alpha(\alpha)|_N$ for some $s \in \{L, R\}$, then $|u| \geq N - M_\alpha$.
3. The symbol $\star$ does not occur in $g|_N$, and there exists a word u such that $g|_N = usI^\alpha(\alpha)|_N$ for some $s \in \{L, R\}$, and $|u| < N - M_\alpha$.
4. $g = u \star I^\alpha(\alpha)$ for some finite word $u \in \{L, R\}^*$, and $|u| < N - M_\alpha$.

In case 1), the hypothesis of Lemma 3.3.4 are satisfied, so there is nothing to prove. Now we turn to the second case. The desired leaves will be the boundary leaves of a certain cylinder containing x. Let $g|_N \in \{L, R\}^\infty$ be the first N letters in the itinerary of x. The cylinder $C(g|_N)$ is a union of open intervals with disjoint closure, with total length 2^{-N}, that contains x. By Proposition 3.2.12, the boundary leaves of the cylinder are of the form $\widetilde{u(i)}\{*_1, *_2\}$, where $u(i)$ are words such that $g|_N = u(i)sI^\alpha(\alpha)|_N$. These boundary leaves clearly form a circular chain around x. It remains to verify that these boundary leaves satisfy the first conclusion of Lemma. Let m be the number of boundary leaves of the cylinder $C(g|_N)$. By Lemma 3.3.6, $m \leq M_\alpha + 1$.

The idea for case 3) is similar. We would like to use the boundary leaves of $C(g|_N)$, but the problem is that not all the boundary leaves are deep enough: if (u_j) denotes the initial subwords of g for which $g = u_j s I^\alpha(\alpha)$, there exists j' such that $|u_{j'}| < N - M_\alpha$. See Figure 3.3d. However, Lemma 3.3.6 guarantees that there can only be one j' for which this holds. It suffices to find another cylinder $C(\hat{g}|_N)$ such that $C(\hat{g}|_N)$ and $C(g|_N)$ share the troublesome shallow boundary leaf $\widetilde{u_{j'}}\{*_1, *_2\}$. The boundary leaves of the closed union $\overline{C(g|_N) \cup C(\hat{g}|_N)}$ will all be deep since the shallow leaf is in the interior and is no longer on the boundary, and we can use these boundary leaves as the leaves in our circular chain.

If $g|_N = u_{j'} s I^\alpha(\alpha)|_N$, let $\hat{g} = u_{j'} \hat{s} I^\alpha(\alpha)|_N$ where $\hat{s} = L$ if $s = R$ and vice versa. Then $C(\hat{g})$ is a cylinder that also has $\widetilde{u_{j'}}\{*_1, *_2\}$ as a boundary leaf. This completes the proof for case 3).

For case 4), we consider the modified itinerary g^L which is equal to $g = I^\alpha(x)$ except that the symbol $\star$ is replaced by the symbol L. The modified itinerary $g^L = uLI^\alpha(\alpha)$ satisfies the hypotheses of case 3) with $t = |u|$, and $\overline{C(g^L|_N)}$ contains x by Proposition 3.2.11, so we can apply the construction of case 3) to g^L to obtained the desired circular chain. $\square$

The existence of these circular chains allows us to pullback the gluings around the main leaf to create gluings on all scales.

Proof of Theorem 3.1.5. Suppose $x \in \mathbb{T}$ and $N \geq 0$. Let $g = I^\alpha(x)$ be the itinerary of x. If the hypotheses of Lemma 3.3.4 are satisfied, we are done. Otherwise, let $u(1), \ldots, u(m)$ be the finite words provided by Lemma 3.3.7, so that the leaves $l_j := \widetilde{u(j)}\{*_1, *_2\}$ form a circular chain around x. Let U be the *interior* of this chain (see Definition 3.3.5).

For each $j = 1, \ldots, m$, the mapping $\widetilde{u(j)}$ maps the two intervals I', I'' (containing $*_1, *_2$) to a pair of neighborhoods $\widetilde{u(j)}I'$ and $\widetilde{u(j)}I''$. Here it is crucial that I' and I'' do not contain any postcritical points.

By backward invariance of $\sim$ and linearity of the $\tilde{u}$, the pair of good gluings between the intervals I_j^- and I_j^+ constructed in Proposition 3.3.3 are mapped via $\widetilde{u(j)}$ to a pair of good gluings adjacent to $\widetilde{u(j)}*_1$ and $\widetilde{u(j)}*_2$. One of the gluings will be in the interior U and the other one will lie in the *exterior* $\mathbb{T}\backslash U$. We will use the gluing that lies in the exterior. The derivative of $\widetilde{u(j)}$ is between 2^{-N} and 2^{-N+M_α}, so the gluing is at scale 2^{-N}.

The collection of these gluings for $j = 1, \ldots, m$ show that $r = 2^{-N}$ is a L-qs-good scale at x, where the constant L does not depend on x and N. $\square$

3.4 Combinatorial semihyperbolicity and concrete semihyperbolicity

So far in this chapter we have worked only with 'abstract' or 'combinatorial' laminations. In this section we relate our work to concrete Julia sets by proving Theorem 3.1.6.

For $c \in \mathbb{C}$ let $p_c(z) = z^2 + c$ and let K_c be its filled Julia set. If K_c is connected there is a unique Riemann map $\varphi_c : \mathbb{D}^* \to \mathbb{C} \backslash K_c$ that fixes ∞ and $\varphi'(\infty) > 0$. If, in addition, K_c is locally connected, then φ_c extends continuously to the boundary $\partial\mathbb{D} \cong \mathbb{T}$. Let $\gamma : \mathbb{T} \to J_c$ be the restriction of this extension to the boundary, where $J_c = \partial K_c$ is the Julia set of p_c. We call γ the Carathéodory loop. The Carathéodory loop induces an equivalence relation $\sim_c$ on $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ where points are identified if they have the same image under γ.

Recall the semiconjugacy of Proposition 3.1.2:

$$p_c \circ \gamma = \gamma \circ h. \tag{3.11}$$

The semiconjugacy implies that $\sim_c$ is closed and invariant in the sense of Section 3.2.1. Now suppose there exists $\alpha \in \mathbb{T}$ that satisifies $\gamma(\alpha) = c$, then $*_1 \sim_c *_2$. We immediately get that $\sim_c$ contains the minimal α-equivalence $\sim_\alpha$.

Recall from Section 3.1 that $c \in \mathbb{C}$ is a *semihyperbolic* parameter if p_c has no parabolic periodic points and if c is not in the closure of its forward orbit.

Our characterization of the topology of $\mathbb{T}/\approx_\alpha$ in terms of itineraries shows that our notion of combinatorial semihyperbolicity (Definition 3.2.5) is equivalent to the notion of semihyperbolicity described above.

Proof of Theorem 3.1.6. For the first direction, suppose c is semihyperbolic and let $\alpha \in \mathbb{T}$ be a landing angle for c so that $\gamma(\alpha) = c$. If $h^t\alpha = \alpha$, then by the semiconjugacy (3.11) we have that $p_c^t \circ \gamma(\alpha) = \gamma(\alpha)$. Since c is not periodic, we conclude that α is not periodic.

From [8, Theorem 1] we get that $\sim_\alpha = \approx_\alpha$ and that $\mathbb{T}/\approx_\alpha$ is homeomorphic to J_c via the map $\gamma : \mathbb{T} \to J_c$. Since c is semihyperbolic we have $c \notin \overline{\bigcup_{t\geq 1} p_c^t(c)}$ in $\mathbb{T}/\approx_\alpha$ (Theorem 3.1.4).

The semiconjugacy (3.11) implies that $[\alpha] \notin \overline{\bigcup_{t\geq 1} h^t([\alpha])}$. It follows from the characterization of the topology of $\mathbb{T}/\approx_\alpha$ in Proposition 3.2.7 that α is combinatorially semihyperbolic.

For the other direction, if α is combinatorially semihyperbolic, then our construction Theorem 3.1.5 together with Theorem 2.5.3 implies that there exists a conformally removable compact set $J \subset \hat{\mathbb{C}}$ such that conformal map $\varphi : \hat{\mathbb{C}}\backslash\mathbb{D} \to \hat{\mathbb{C}}\backslash J$ solves the welding problem $\approx_\alpha$, meaning $\varphi(e^{2\pi i x)}) = \varphi(e^{2\pi i y)}) \iff x \approx_\alpha y$. Suppose we chosen J so that φ satisfies the normalizations $\varphi(e^{2\pi i *_1}) = 0$, $\varphi(\infty) = \infty$, and $\varphi(z) = z + O(1)$ as $z \to \infty$. The $x \mapsto -x$ symmetry of $\approx_\alpha$ implies that φ is odd, so actually $\varphi(z) = z + o(1)$.

We will show that J is the Julia set of $z \mapsto z^2 + c$, where $c = \varphi(e^{2\pi i\alpha})$.

Let $\tilde{h}(z) = z^2$ be the extension of the angle doubling map to the exterior of the unit disk, $\tilde{h} : \hat{\mathbb{C}}\backslash\mathbb{D} \to \hat{\mathbb{C}}\backslash\mathbb{D}$, and let $p = \varphi \circ \tilde{h} \circ \varphi^{-1}$ be its conjugate on $\hat{\mathbb{C}}\backslash J$. Then p is holomorphic on $\hat{\mathbb{C}}\backslash J$, and backward invariance of $\approx_\alpha$ implies that p extends continuously to $\hat{\mathbb{C}}$ to a topological degree two branched cover with two critical values, at ∞ and at $c = \varphi(e^{2\pi i\alpha})$. The normalization of φ implies that $p(z) = z^2 + O(1)$ as $z \to \infty$.

We would like to conclude from removability of J that p is holomorphic on $\hat{\mathbb{C}}$ and is hence a polynomial. However, p is not a homeomorphism so we cannot apply conformal removability directly. Instead, we will consider a lift of p.

Consider the map $p_c(z) = z^2 + c$ where $c = \varphi(e^{2\pi i\alpha})$. It is a double sheeted cover of $\mathbb{C}\backslash\{c\}$ by $\mathbb{C}\backslash\{0\}$. On the other hand, $p : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{c\}$ is also a double sheeted cover. Let π be a homeomorphism $\pi : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ such that $p = p_c \circ \pi$. Note that if $B \subset \mathbb{C}$ is a ball around 0, then $p_c^{-1}(p(B))$ is also a ball around 0. Therefore we can extend π to a homeomorphism $\mathbb{C} \to \mathbb{C}$ by defining $\pi(0) = 0$. Since $p(z) = z^2 + O(1)$ and $p_c(z) = z^2 + O(1)$ as $z \to \infty$, it follows that $\pi(z) = z + o(1)$ as $z \to \infty$.

On the other hand, the holomorphicity of p on $\mathbb{C}\backslash J$, and the holomorphicity of p_c, implies that π is holomorphic on $\mathbb{C}\backslash J$. By removability of J we conclude that $\pi : \mathbb{C} \to \mathbb{C}$ is the identity.

We have thus shown that $p_c = p$ and it is clear that J is the filled Julia set of p. We have found a polynomial p_c for which $\mathbb{T}/\approx_\alpha \cong J_c$. It remains to show that c is semihyperbolic.

Since (we have just shown that) γ is a homeomorphism between $(\mathbb{T}/\approx_\alpha, h)$ and (J_c, p_c), it suffices to show that $[\alpha] \notin \overline{\bigcup_{t\geq 1} h^t([\alpha])}$. By Proposition 3.2.7, we need to show that for all $t \geq 1$, $\sigma^t I^\alpha(\alpha)$ is not $(2M_\alpha+1)$-close to $I^\alpha(\alpha)$. Suppose $I_\alpha(h^t\alpha)$ and $I_\alpha(\alpha)$ are $2M_\alpha+1$ close for some t, then we can write $I_\alpha(\alpha)|_{2M_\alpha+1} = usI_\alpha(\alpha)|_{2M_\alpha+1}$ and $I_\alpha(h^t\alpha)|_{2M_\alpha+1} = us'I_\alpha(\alpha)|_{2M_\alpha+1}$ for some finite word $u \in \{L, R\}^*$ and some $s, s' \in \{L, R\}$. If $|u| > M_\alpha$ then this shows that $I_\alpha(\alpha)|_{M_\alpha} = u|_{M_\alpha} = k_\alpha(h^t\alpha)|_{M_\alpha}$, and this violates combinatorial semihyperbolicity of α. On the other hand if $|u| \leq M_\alpha$, then by applying the shift $\sigma^{|u|+1}$ to the equality $I_\alpha(\alpha)|_{2M_\alpha+1} = usI_\alpha(\alpha)|_{2M_\alpha+1}$ shows that $I_\alpha(h^{|u|+1}\alpha)|_{2M_\alpha-|u|} = I_\alpha(\alpha)|_{2M_\alpha-|u|}$. Since $2M_\alpha - |u| \geq M_\alpha$, this again violates combinatorial semihyperbolicity. $\square$

Chapter 4

CONFORMAL WELDING OF THE CRT

In this chapter, we prove Theorems 1.3.1 and 1.3.2 on the welding of the Brownian lamination and the scaling limit of large Shabat trees. Figure 1.4 shows the solution to the welding problem for the lamination corresponding a regular tree of depth $n = 11$, where each (half)-edge has the same harmonic measure with respect to infinity. It can be shown that as $n \to \infty$, the diameter of the middle edges stays bounded below. This implies that the associated conformal welding maps f_n cannot converge uniformly on $\overline{\mathbb{D}}$, as each edge is the image of an arc of size $\frac{1}{2}(3 \cdot 2^n - 3)^{-1}$ under f_n. However, along subsequences, the f_n do converge locally uniformly to a conformal limit (we believe that the limit exists and that the limit is a conformal map onto a Jordan domain). An easier example of the same phenomenon is the conformal map $z \mapsto (z^n + z^{-n} + 2)^{1/n}$ onto the complement of a star with n edges; these map clearly do not converge uniformly on $\overline{\mathbb{D}}$, however they converge locally uniformly to the identity on $\mathbb{D}^*$.

These examples illustrate the main difficulty in the proof of 1.3.2. To deal with this, we will show that the Brownian lamination satisfies the good scales condition of Theorem 2.5.3 (although it does not satisfy the qs-good scale condition).

The strategy is to construct candidate annuli $\mathcal{A}_l$ (for $l = 1, 2, \dots$) in terms of ω_l where, roughly speaking, ω_l is the part of the excursion S lying between height λ^l and λ^{l+1}. The conditions of Theorem 2.5.1 that ensure the nondegeneracy of $\mathcal{A}_l$ translate into five simple conditions $\mathrm{Good}_{1,2,3,4,5}$ on ω_l, see Section 4.1.2, and its discrete counterpart, Section 4.2.3. We will define ω_l in such a way (see Sections 4.1.2 and 4.3.1) that $(\omega_l)_{l \geq 1}$ is a Markov chain. This allows us to use the standard large deviations framework (Theorem B.0.3) to show that, with very high probability, the density of the set of indices l for which $\omega_l \in \mathrm{Good}_{1,2,3,4,5}$ is greater than $\frac{1}{2}$.

The crux of the proof is a long computation (Lemma 4.3.4) showing that a certain function on the state space decreases sharply in expectation when the Markov chain transitions from a state that is not in $\mathrm{Good}_{1,2,3,4,5}$.

Section 4.1 contains a sketch of a proof of Theorem 1.3.1 along these lines. However, we will not give all the details because the same strategy also gives a proof of the stronger Theorem 1.3.2, which we prove in detail in Section 4.2. For the convenience of the reader, we give a self-contained presentation of the large deviations estimate that we need in Appendix B. Some technical estimates for discrete random walks needed for Section 4.2 are collected in Appendix A.

4.1 Proof sketch for Theorem 1.3.1

Fix a standard Brownian excursion $\mathbb{e} : [0,1] \to [0,\infty)$. We would like to employ Theorem 2.5.3 and show that almost surely, for every $x \in \mathbb{T}$ and every $n \geq 1$, at least half of the scales $2^{-1}, 2^{-2}, \ldots, 2^{-n}$ are good scales. By rotation invariance of the CRT, it suffices to consider $x = 0$ and show that the probability of not having $n/2$ good scales decays exponentially faster than 2^{-n}.

4.1.1 Decomposition of Brownian excursion

The good chains at each scale will be obtained by considering a decomposition of $\mathbb{e}$ into excursions away from H_l that reach height H_{l+1}, where essentially $H_l = \lambda^{l-n}$ for some fixed $\lambda > 1$ and $1 \leq l \leq n$. For ease of notation, we fix j and write $h_1 = H_l, h_2 = H_{l+1}$ and so on in our description of the decomposition below, see Figure 4.1.

Let $\mathcal{X} = \{t : \mathbb{e}(t) = h_1\}$. Consider those connected components U of $[0,1]\backslash\mathcal{X}$ on which $\mathbb{e}$ is an excursion that reaches level h_2, that is $\mathbb{e}|_U \geq h_1$ and $\sup \mathbb{e}|_U \geq h_2$. Suppose there are k such components $U_1, ..., U_k$ (by continuity there are finitely many of these intervals). Then there are $k+1$ components $U_{k+1}, ..., U_{2k+1}$ of the complement $[0,1] \setminus \bigcup_j U_j$. Notice that

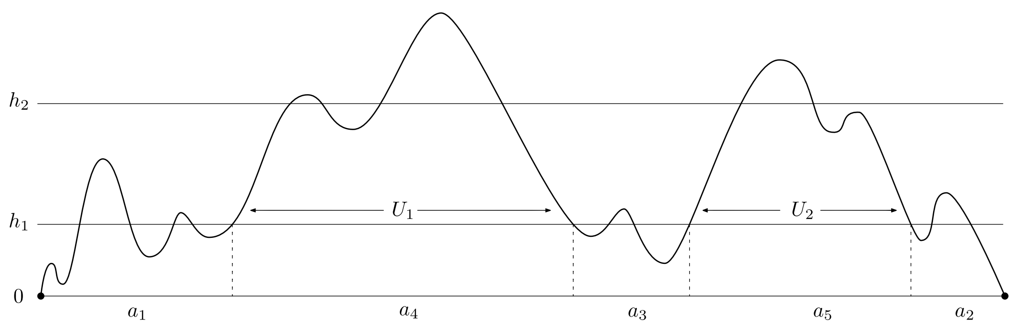


Figure 4.1: The decomposition of a Brownian excursion with respect to heights h_1 and h_2. Here, there are $k = 2$ excursions from height h_1 to height h_2, over the intervals U_1 and U_2. The lengths of the intervals in this decomposition are given by the a_i. The indexing is always chosen so that a_1 and a_2 are the lengths of the intervals on the end, then $a_3, \ldots, a_{k+1}$ are the lengths of the intervals in between the excursion intervals, and finally $a_{k+2}, \ldots, a_{2k+1}$ are the lengths of the excursion intervals themselves.

- Conditioned on the leftmost interval, the law of $\mathbb{e}$ on that interval is that of a Brownian meander conditioned on ending at h_1 and staying below h_2. Similarly, the conditional law of $\mathbb{e}$ on the rightmost interval is that of a time-reversal of such a meander. (A Brownian meander is a Brownian motion starting at 0 conditioned to be positive).

- On the U_j with $1 \leq j \leq k$, the conditional law of $\mathbb{e} - h_1$ is that of an excursion conditioned to reach height $h_2 - h_1$.

- On the remaining U_j, the conditional law of $\mathbb{e}$ is that of a Brownian bridge from h_1 to h_1, conditioned to stay between 0 and h_2.

If $k \geq 1$, the lengths of the intervals U_j can be viewed as a $(2k+1)-$ dimensional vector $\mathbf{a} = (a_1, \ldots, a_{2k+1})$. Re-label the indices so that

- a_1 and a_2 denote the lengths of the left- and rightmost interval.

- $a_3, \ldots, a_{k+1}$ denotes the lengths of the $k-1$ bridges, in left to right order.

- $a_{k+2}, \dots, a_{2k+1}$ denotes the lengths of the k excursions, in left to right order.

It is possible to write down an explicit expression for the density of the random variable $\mathbf{a}$, see Proposition 4.2.2 below for the analog in the discrete setting.

Remark 4.1.1. Here are some intuitive statements about $\mathbf{a}$. By Brownian scaling we may assume $h_1 = 1$ and $h_2 = \lambda h_1$ for some fixed $\lambda > 0$. Let $W > 0$ be the length of the excursion after this rescaling.

- If W is very small, then $k = 0$ with high probability.
- k has exponential tails, uniformly as $W \to \infty$.
- It is very unlikely for a Brownian bridge to stay in an interval of size h_1 over a time period much longer than h_1^2. It follows that it is very unlikely for the lengths $a_3, \dots, a_{k+1}$ to be much longer than h_1^2.
- For $W/h_1^2 \gg 1$, it is likely that most of the mass of the interval $[0, W]$ goes to a single a_i: For example, it is much more likely that there is an a_i with $a_i \approx W_i$ than it is to have a_i and a_j with $a_i \approx a_j \approx W/2$.

4.1.2 Constructing chains for the Brownian excursion

We now explain how the decomposition defined in the previous section can be used to construct good chains (see Section 2.5 for definitions).

Fix N large and for $l = 0, 1, 2, \dots$ consider the geometric sequence of scales $H_0 = 0$ and

$$H_{l+1} = H_l + \lambda^{-N} \lambda^l$$

so that

$$H_{l+2} - H_{l+1} = \lambda(H_{l+1} - H_l).$$

Let $\mathbb{e} : [0,1] \to [0, \infty)$ be an excursion.

Fix a 'scale' $h_0 = H_l, h_1 = H_{l+1}, h_2 = H_{l+2}$ and denote $H_{l+.5} = h_{1.5}$ the point in between h_1 and h_2 satisfying

$$\frac{h_2 - h_{1.5}}{h_{1.5} - h_1} = \Lambda$$

where Λ is a large parameter that is determined later. Fix an excursion interval $U_j \subset \mathbb{T}$, $1 \leq j \leq k$, so that by our definition $\inf \mathbb{e}|_{U_j} = h_1$ and $\sup \mathbb{e}|_{U_j} \geq h_2$. Let $\tau = \tau_j = \inf\{t \in U_j : \mathbb{e}|_{U_j}(t) = h_2\}$ and $\tilde{\tau} = \sup\{t \in U_j : \mathbb{e}|_{U_j}(t) = h_2\}$ be the first and last times respectively that $\mathbb{e}|_{U_j}$ visits h_2. Let $\tau^- = \sup\{t \in U_j : t < \tau, \mathbb{e}(t) = h_{1.5}\}$ be the last time that $\mathbb{e}|_{U_j}$ visits $h_{1.5}$ before visiting h_2. Let $\tau^+ = \inf\{t \in U_j : t > \tilde{\tau}, \mathbb{e}(t) = h_{1.5}\}$ be the first time that $\mathbb{e}|_{U_j}$ visits $h_{1.5}$ after visiting h_2 for the last time, see Figure 4.4 below.

The endpoints of U_j, call them θ^- and θ^+, are equivalent, so if τ^- and τ^+ are equivalent, the pair of intervals $\mathcal{C}^{(l,j)} := ([\theta^-, \tau^-], [\tau^+, \theta^+])$ form a chain link. Define the chain $\mathcal{C}^{(l)}$ as the sequence of chain links $\mathcal{C}^{(l,j)}$ for $j = 1, \ldots, k$, see Figure 4.2.

The following conditions $\mathrm{Good}_1, \mathrm{Good}_2, ..., \mathrm{Good}_5$ are translations of the conditions of Section 2.5, and they guarantee the desired lower bound on $\mathrm{mod}(\Gamma(\mathcal{C}^{(l)}))$. They all involve the parameter $L > 0$, where larger L corresponds to less restrictive conditions. See Section 4.2.3 for the detailed definitions on these conditions in the discrete setting.

We say that $S|_{U_j} \in \mathrm{good}_1(h_1, h_2)$ if the restriction of the jth excursion to $[\tau^-, \tau^+]$ does not dip below height $h_{1.5}$, so that τ^- and τ^+ are identified via $\sim$ (Figure 4.2), and we say that $S \in \mathrm{Good}_1(h_1, h_2)$ if $S|_{U_j} \in \mathrm{good}_1(h_1, h_2)$ holds for all $1 \leq j \leq k_l$.

If Good_1 holds then by the discussion above, we get a well defined chain link $\mathcal{C}^{(l)}$.

The remaining conditions $\mathrm{Good}_{2,3,4,5}$ ensure that this chain link satisfies the conditions of Theorem 2.5.1.

- We say that $S|_{U_j} \in \mathrm{good}_2(h_1, h_2)$ if the diameters of the two intervals in the chain link $C^{(l,j)}$ are comparable to h_1^{-2}, and we say that $S \in \mathrm{Good}_2$ if $S|_{U_j} \in \mathrm{Good}_1(h_1, h_2)$ for all $j = 1, \ldots, k$.

- We say that $S|_{U_j} \in \mathrm{good}_3(h_1, h_2)$ if the excursion is Hölder on the chain link intervals of $C^{(j)}$. This corresponds to Condition 2 in Section 2.5.

- We say that $S \in \mathrm{Good}_3(h_1, h_2)$ if $S|_{U_j} \in \mathrm{good}_3(h_1, h_2)$ for all $j = 1, \ldots, k$.

- The Good_4 condition is satisfied if the total length of the non-excursion intervals is comparable to h_1^{-2}. This, together with the Good_1 condition, corresponds to Condition 1 in Section 2.5.

- The Good_5 condition is satisfied if the number of excursion intervals is bounded by L, that is $k \leq L$. This corresponds to Condition 3 in Section 2.5.

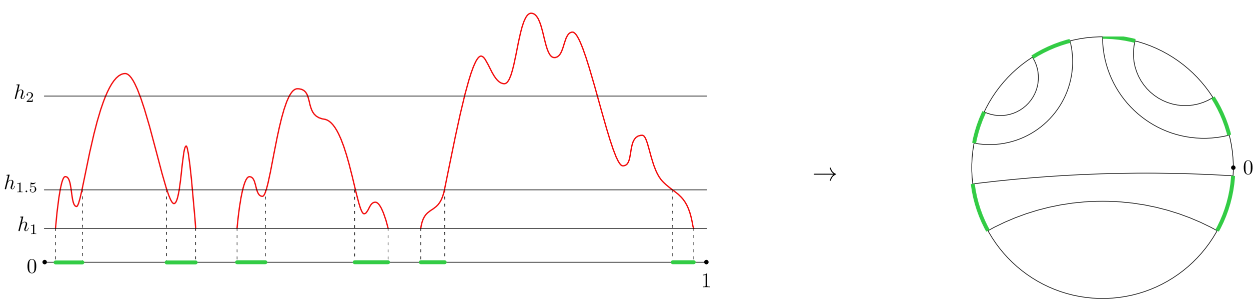


Figure 4.2: Left: we have a Brownian excursion which has $k = 3$ excursions from $h_1 = \lambda^l h$ to $h_2 = \lambda^{l+1} h$. The rest of the excursion is irrelevant and not shown here. The conditions $\mathrm{Good}_1(l, j)$ are satisfied for $j = 1, 2, 3$, and the resulting chain links are highlighted on the x-axis.

Every scale that satisfies the Good conditions gives rise to a good chain in Theorem 2.5.3. Thus the proof of Theorem 1.3.1 reduces to showing that the Good conditions hold at many scales l. If the scales were independent, it would suffice to estimate the probability that a given scale satisfies the Good conditions. Since the scales are not independent, we have to work a little harder. We analyze them via a discrete time Markov exploration process ω_l, where ω_l consists of the following information:

- The excursion intervals U_j of $\mathbb{e}$ from H_l to H_{l+1} (described in Section 4.1.1).

- The excursion intervals V_i from H_{l+1} to H_{l+2}

- The restriction of $\mathbb{e}$ to the U_j, modulo the restriction onto the V_i. In other words, we keep track of what happens on the U_j, but we 'forget' what happens on the V_i intervals.

If $\mathbb{e}$ is distributed as a Brownian excursion, then the sequence $(l, \omega_l)_{l \geq 0}$ is a Markov chain. Some aspects of this Markov chain can be computed explicitly. For example, let $\mathbf{a}(\omega_l)$ denote the sequence of lengths of the excursion intervals V_i above. This is also a Markov chain and its transition probabilities can be computed explicitly. These transition probabilities will respect the Brownian scaling. See the next section for the details in the discrete setting.

Each of the Good_i conditions can be identified with a certain subset of the state space of this Markov chain, and the following large deviation estimate (proved in Appendix B) can be applied.

Theorem 4.1.2. *Let ω_l be a Markov chain on state space Ω with transition densities $\pi(x, dy)$. Let $A \subset \Omega$ and suppose $u : \Omega \to [1, \infty)$ is a function with*

$$\tilde{\lambda}_u(x) := \log\left(\frac{u(x)}{\int u(y)\pi(x, dy)}\right) \geq 0. \tag{4.1}$$

Then for each $\epsilon > 0$,

$$\mathbb{P}\left(\frac{1}{n}\,\left|\{k : \omega_k \in A\}\right| \geq \epsilon\right) \leq \mathbb{E}u(\omega_1)\exp\left(-n\epsilon \inf_{\omega \in A} \tilde{\lambda}_u(\omega)\right). \tag{4.2}$$

It remains to construct a test function u such that (4.1) is satisfied and such that $\inf_{\omega \notin \text{Good}} \tilde{\lambda}_u(\omega)$ is large. For the sake of exposition, we first describe how to create a test function that gives us a bound for the Good_5 condition.

In our exploration process, each excursion interval splits into multiple excursion intervals, independently of the other excursion intervals. Generically, there will be one large excursion interval which is sustained from level to level, and occasionally this excursion interval will have a few child intervals of short length. These shorter excursion intervals will tend to not have too many children (see Proposition 4.3.2), and so it is plausible that at most levels, the Good_5 condition is satisfied. To understand the choice of test function, it is helpful to use subcritical Galton-Watson branching with immigration as a simplified toy model of the process. This is the sequence of random variables $Z_0, Z_1, Z_2, \dots$ where $Z_0 = 0$ and

$$Z_{n+1} = 1 + \sum_{i=1}^{Z_n} \Xi_i,$$

and the Ξ_i are i.i.d. random variables of mean strictly less than 1, supported on the nonnegative integers. The immigrant (the 1+ term) plays the role of the large excursion interval. The (Z_i) form a Markov chain on the state space $\{0, 1, 2, \dots\}$. In this case, the test function $u(\omega) := \zeta^\omega$ for some appropriately chosen constant $\zeta > 1$ can be used in Theorem 4.1.2 to get large deviations upper bounds on the density of generations for which Z_n is large. Indeed, the fact that each node has its children independently allows the exponent in the right hand side of (4.2) to be bounded.

One aspect in which the Markov chain on Brownian excursions differs from the simplified toy model is that the 'nodes' in the Brownian excursion exploration process are themselves Brownian excursions. In particular, they have different lengths, which affects their offspring distribution.

This can be modeled by a multi-type Galton-Watson process: each 'type' has its own offspring distribution (which not only contains information about the number of offspring, but also the type of offspring). In this setting, the test function has to take into account the different types: $u(Z_n) = \prod_{i=1}^{Z_n} \zeta(\text{Type}(Z_i))$ where ζ is now a real valued function of types.

For the exploration process of excursion intervals, the 'type' of the excursion interval is the (Brownian scaled) length $\beta > 0$ of the excursion interval, and it turns out (c.f. Lemma 4.3.5) that $\zeta(\beta) = 2 + \beta^{1/4}$ works.

That is, if $\lambda > 1$ is sufficiently large, then it can be shown that the following test function $u = V$ satisfies (4.1)

$$V(l, \omega) = \prod_{i=k+2}^{2k+1} (2 + (a_i g_l^{-2})^{1/4}),$$

where $g_l = H_{l+2} - H_{l+1}$ and the a_i are the lengths of the excursion intervals from H_{l+1} to H_{l+2}. Note that the state space of our markov chain is the space of pairs (l, ω). Furthermore, if $A = \{\omega : \omega \text{ has more than } L \text{ excursion intervals}\}$ (i.e. the complement of the Good_5 states), then by choosing L sufficiently large, we can make $\inf_{\omega \in A} \tilde{\lambda}_u(\omega)$ arbitrarily large.

Now we write down the more complicated test function that can be used to get large deviations for all the $\text{Good}_{1,2,3,4,5}$ conditions. Calculation shows that for suitable parameters $q, \lambda > 1$ large, and $s > 1$, $W_0 > 0$ small, the following function has the desired properties:

$$V(l, \omega) = s^{g_l^{-2} - \sum_{i=1}^{k_l} \alpha_i} \prod_{i=1}^{k_l} q^{\mathbb{1}(\alpha_i \le W_0)} (2 + \alpha_i^{1/4})^{1/2} \prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j} \notin \text{good}_{1,2,3}(H_l, H_{l+1}))}, \tag{4.3}$$

where for $i = 1, \ldots, k$, the $\alpha_i = \mathbf{a}(\omega_l)_{i+k+1} g_l^{-2}$ are the scaled lengths of the excursions intervals from H_{l+1} to H_{l+2}.

This definition depends on several different parameters, some of which have already been introduced. We summarize them here for the reader's convenience. All these constants except for s and W_0 will be taken to be 'large'.

1. $q > 1$ is a penalty factor for violating the $\text{good}_{1,2,3}$ condition and also a penalty for any excursion intervals that are too short. It will turn out that we need to take $q \asymp \lambda^{20}$.

2. $L > 1$ is a parameter that determines how restrictive the $\text{Good}_{1,2,3,4,5}$ conditions are.

3. $W_0 > 0$ is a parameter that determines what constitutes a 'short' excursion interval. We need to penalize short excursion intervals so that we can ensure the $\text{good}_{1,2,3}$ conditions are satisfied (see the hypotheses of Proposition 4.3.3).

4. $s > 1$ is a penalty factor for violating the Good_4 condition.

5. $\lambda \ge 2$ is the step size for the Markov exploration process.

6. $\Lambda \ge 2$ determines the relative distances between $H_l, H_{l+0.5}$, and H_{l+1}. Changing this parameter affects the definition of the $\text{good}_{1,2,3}$ conditions.

In (4.3), the factor $s^{g_l^{-2}-\sum_{i=1}^{k_l}\alpha_i}$ penalizes states for which much of the interval $[0,1]$ is taken up by non-excursion-intervals. Whenever the Good_4 condition is violated, this factor is large. However, this tends to decrease under iteration of the Markov chain due to the extra factor of λ^{-2} from rescaling.

Using the explicit equations for the transition probabilities of the Markov chain, it can be shown that the test function (4.3) has the desired properties. We will not present the proof here. Instead, we will prove the analogous result (Lemma 4.3.4) for the discrete approximations to the Brownian excursion.

4.2 The exploration process for large finite trees

In this section we present the details of the proof of Theorem 1.3.2, following the strategy of the proof of Theorem 1.3.1 outlined in the previous section. As Theorem 1.3.2 implies Theorem 1.3.1, this also concludes a detailed proof of Theorem 1.3.1. Before adopting the decomposition described in Section 4.1.1 to the setting of random walks, we collect some notation and terminology.

4.2.1 Notation and terminology

A *(Bernoulli) walk* of length n is a map $S : \{0, \dots, n\} \to \mathbb{Z}$ such that $S_{i+1} - S_i \in \{-1, 1\}$ for $i \geq 1$. For the rest of this paper we will assume that Bernoulli walks are defined on the whole interval $[0, n]$ by requiring that the walk is linear of slope 1 in between the integer points.

For $a \in \mathbb{Z}$ denote $\mathbf{W}_n(a)$ denote the collection of walks of length n with $S_0 = a$. Let $\mathbf{W}_n(a \to b) \subset \mathbf{W}_n(a)$ denote the set of walks S with $S_n = b$, so that $|\mathbf{W}_n(a)| = 2^n$ and

$$|\mathbf{W}_n(a \to b)| = \binom{n}{n/2 - (b-a)/2}. \tag{4.4}$$

For this formula to be true when n is odd, we abide by the convention that binomial coefficients with noninteger arguments are equal to zero.

Let $\mathbf{E}_n(a) \subset \mathbf{W}_n(a \to a)$ denote the collection of *excursions away from a of length n*, namely walks with $S_0 = S_n = a$, and $S_i \geq a$ for all i. Note that $\mathbf{E}_n(0)$ is the collection of

Dyck paths of length n, and recall that $|\mathbf{E}_n(0)|$ is given by the Catalan number $\frac{1}{n/2+1}\binom{n}{n/2}$ (this can be deduced by taking $a = n+1$ and $g = 1$ in Corollary A.0.2).

Fix an excursion S from 0. As in the previous section, we will consider the excursions of S away from h_1 that exceed level h_2, where $0 = h_0 < h_1 < h_2$. As before, this naturally leads us to consider the partition of $[0, n]$ into disjoint (except for their endpoints) closed intervals, where the restriction of S onto each part corresponds to one of the following three types:

- For $n \geq 2$, let $\mathbf{W}_n^{\uparrow}(a \to b) \subset \mathbf{W}_n(a \to b)$ denote the walks which 'approach a and b from below', that is

  $$\mathbf{W}_n^{\uparrow}(a \to b) = \{S \in \mathbf{W}_n(a \to b) : S_1 = a-1 \text{ and } S_{n-1} = b-1\}.$$

  This definition is needed to guarantee uniqueness of the decomposition, Proposition 4.2.2.

  Note the natural bijection $\mathbf{W}_n^{\uparrow}(a \to b) \cong \mathbf{W}_{n-2}(a-1 \to b-1)$ which together with (4.4) yields $|\mathbf{W}_n^{\uparrow}(a \to b)| = \binom{n-2}{n/2-1-(b-a)/2}$. For $c < d$, let $\mathbf{W}_n^{\uparrow}(a \to b; c \leq \min < \max \leq d)$ be the set of walks S in $\mathbf{W}_n^{\uparrow}(a \to b)$ for which $c \leq S \leq d$.

- Let $\mathbf{Z}_n(a \uparrow b) \subset \mathbf{W}_n(a \to b)$ denote the walks that stay above the left endpoint a and 'approach the right from below', that is

  $$\mathbf{Z}_n(a \uparrow b) = \{S \in \mathbf{W}_n(a \to b) : S \geq a \text{ and } S_{n-1} = b-1\}.$$

  Similarly, let $\mathbf{Z}_n(a \downarrow b) \subset \mathbf{W}_n(a \to b)$ denote the walks that stay above the right endpoint and 'approach the left from below', that is

  $$\mathbf{Z}_n(a \downarrow b) = \{S \in \mathbf{W}_n(a \to b) : S \geq b \text{ and } S_1 = a-1\}.$$

  Notice that there is a natural bijection $\mathbf{Z}_n(a \uparrow b) \cong \mathbf{Z}_n(b \downarrow a)$ by time reversal. Corollary A.0.2 shows that $|\mathbf{Z}_w(a \uparrow b)| = \frac{b-a}{w}\binom{w}{\frac{w}{2}+\frac{b-a}{2}}$.

- For $b > a$, let $\mathbf{E}_n(a; \max \geq b)$ be the collection of excursions in $\mathbf{E}_n(a)$ with maximum greater than or equal to b.

We will often need to consider the uniform probability measure on these spaces of walks. We will use a subscript to denote the probability measure in question, and the variable S to denote the random variable; for instance $\mathbb{P}_{\mathbf{W}_n(a\to b)}(S \in \cdot)$ denotes the uniform probability distribution on $\mathbf{W}_n(a \to b)$. Thus for $A \subset \mathbf{W}_n(a \to b)$ we have $\mathbb{P}_{\mathbf{W}_n(a\to b)}(S \in A) = \frac{|A|}{|\mathbf{W}_n(a\to b)|}$.

In what follows, we will frequently deal with walks and excursions that are defined on intervals $I = [u, v]$ instead of on $[0, n]$. Therefore it will be convenient to use the notation $\mathbf{W}_I, \mathbf{E}_I$, and so on, with the obvious meaning. If there is no subscript, the union over all intervals (with integer endpoints) is taken. For example, $\mathbf{W}(a) = \bigcup_I \mathbf{W}_I(a)$.

4.2.2 Excursion decomposition of Dyck paths

We begin by illustrating the decomposition with an example. The description of the decomposition in general follows later. Let $n = 28, h_1 = 2$ and $h_2 = 4$. Consider the walk

$$S = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 0 & 1 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

where $S \in \mathbf{E}_{28}(0 \to 0)$. For each column in the above table, the bottom entry is the value of S at the time given in the top entry. See Figure 4.3.

We can write S as the following concatenation of walks:

$$S = B_1 A_1 B_2 A_2 B_3$$

where

$$B_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 1 & 2 & 3 & 2 & 1 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & 4 & 5 & 4 & 3 & 2 & 3 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 2 & 1 & 2 & 3 & 2 & 1 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 22 & 23 & 24 & 25 & 26 \\ 2 & 3 & 4 & 3 & 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 26 & 27 & 28 \\ 2 & 1 & 0 \end{bmatrix}.$$

The key point here is that $B_1 \in \mathbf{Z}(0 \uparrow h_1, \max S < h_2)$, $B_2 \in \mathbf{W}^{\uparrow}(h_1 \to h_1, 0 \le \min < \max < h_2)$, and $A_1, A_2 \in \mathbf{E}(h_1 \to h_1, \max \ge h_2)$ and $B_3 \in \mathbf{Z}(h_1 \downarrow 0, \max S < h_2)$. The decomposition $S = B_1 A_1 B_2 A_2 B_3$ naturally induces the desired partition $\{[0, 8], [8, 16], [16, 22], [22, 26], [26, 28]\}$ of $[0, 28]$.

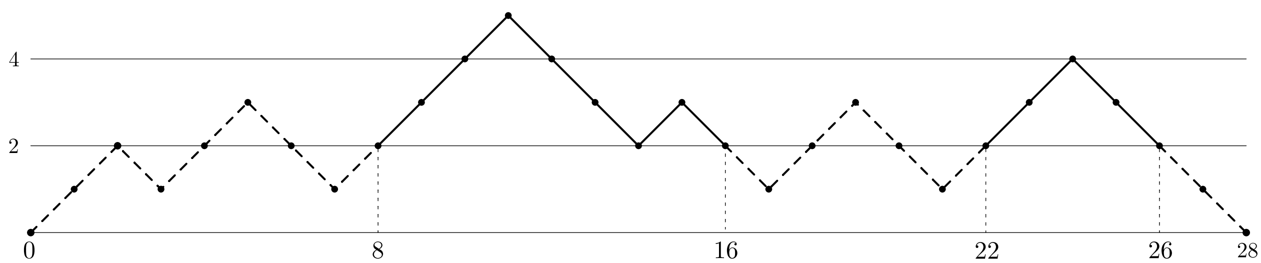


Figure 4.3: An excursion in $\mathbf{E}_{28}(0 \to h_1)$ and its decomposition into concatenations of 5 smaller excursions and bridge walks. Here $h_1 = 2$, $h_2 = 4$ and $n = 28$. The first and last walks in this decomposition are walks of type $\mathbf{Z}(0 \uparrow h_1, \max < h_2)$ and $\mathbf{Z}(h_1 \downarrow 0, \max < h_2)$ respectively. The second and fourth walks are of type $\mathbf{E}(h_1 \to h_1, \max \geq h_2)$. The third walk is of type $\mathbf{W}_n^{\uparrow}(h_1 \to h_1, 0 \leq \min < \max < h_2)$.

In general, fix integers $h_2 > h_1 > h_0 = 0$ and suppose $S \in \mathbf{E}_n(0, \max \geq h_2)$ is an excursion that reaches level h_2. Let $k = k(S) \geq 1$ be the number of excursions away from h_1 that reach level h_2. Then we can decompose S into a concatenation of walks

$$S = Z_1 E_1 B_1 E_2 B_2 \cdots B_{k-1} E_k Z_2$$

where

- $Z_1 \in \mathbf{Z}(0 \uparrow h_1, \max < h_2)$ and $Z_2 \in \mathbf{Z}(h_1 \downarrow 0, \max < h_2)$
- For $i = 1, \ldots, k-1$, $B_i \in \mathbf{W}^{\uparrow}(h_1 \to h_1, 0 \leq \min < \max < h_2)$.
- For $i = 1, \ldots, k$, $E_i \in \mathbf{E}(h_1, \max \geq h_2)$.

Definition 4.2.1. We will refer to the walks in the decomposition as $[0 \uparrow h_1 \uparrow h_2]$-ends, $[0 \uparrow h_1 \uparrow h_2]$-bridges and $[0 \uparrow h_1 \uparrow h_2]$-excursions (which we often abbreviate as $[h_1 \uparrow h_2]$-excursions), respectively, of S. We will also call the intervals in this decomposition the $[0 \uparrow h_1 \uparrow h_2]$-end intervals of S and so on. We denote $\mathbf{a}(S) = \mathbf{a}_{[0\uparrow h_1 \uparrow h_2]}(S) = (a_1, \ldots, a_{2k+1}) \in$

$\mathbb{Z}_{\geq 0}^{2k+1}$ the vector of lengths of the intervals in this decomposition, and will always choose the indexing of the a_i as in Section 4.1.1:

- a_1 and a_2 are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-end intervals.
- $a_3, \ldots, a_{k+1}$ are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-bridge intervals, in left to right order.
- $a_{k+2}, \ldots, a_{2k+1}$ are the lengths of the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals, in left to right order.

Similarly, we define the $[H_1 \uparrow H_2 \uparrow H_3]$-decomposition of a walk S by translation as the $[0 \uparrow H_2 - H_1 \uparrow H_3 - H_1]$-decomposition of $S - H_1$, and we often abbreviate $[H_1 \uparrow H_2 \uparrow H_3]$-excursion intervals to $[H_2 \uparrow H_3]$-excursion intervals.

The indexing of the a_i above is consistent with the notation in the following simple consequence of the uniqueness of the above decomposition:

Proposition 4.2.2. *Fix integers $h_2 > h_1 > a$ and $n \geq 2$. There is a bijection*

$$\begin{aligned}\mathbf{E}_n(0) \cong\ & \mathbf{E}_n(0, \max < h_2) \sqcup \\ & \sqcup \bigsqcup_{k=1}^{\infty} \bigsqcup_{a_1 + \cdots + a_{2k+1} = n} \mathbf{Z}_{a_1}(0 \uparrow h_1, \max < h_2) \times \mathbf{Z}_{a_2}(h_1 \downarrow 0, \max < h_2) \times \\ & \times \prod_{i=3}^{k+1} \mathbf{W}^{\uparrow}_{a_i}(h_1 \to h_1, 0 \leq \min < \max < h_2) \\ & \times \prod_{i=k+2}^{2k+1} \mathbf{E}_{a_i}(h_1, \max \geq h_2),\end{aligned} \tag{4.5}$$

where the second disjoint union is taken over positive integers $a_i \geq 0$.

4.2.3 Chains and conditions for large modulus

In this section we show how the decomposition introduced in the previous Section 4.2.2 naturally leads to chains in the sense of Section 2.5. We then identify several conditions that the decomposition at a given level must satisfy for the corresponding chain to be good. In the subsequent sections we will show that these conditions are satisfied at many scales.

Fix $0 < h_1 < h_2$ integer. Let $S \in \mathbf{E}_n(0, \max \geq h_2)$ be an excursion that reaches height h_2. Let $U_1, \ldots, U_k$ be the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals of S so that all $S|_{U_j} \in \mathbf{E}(h_1, \max \geq h_2)$.

We now describe these various conditions as subsets of $\mathbf{E}(h_1)$, denoted by $\mathrm{Good}_i(h_1, h_2)$ where $1 \leq i \leq 5$. They involve a parameter $L > 1$ where larger choices of L make the conditions less restrictive. In what follows, let $\Lambda > 1$ be an integer and let

$$h_{1.5} = h_1 + \lfloor (h_2 - h_1)/\Lambda \rfloor.$$

The first three conditions are regularity conditions that each of the $[h_1 \uparrow h_2]$-excursions have to satisfy individually, $S \in \mathrm{Good}_{1,2,3}(h_1, h_2)$ if and only if for all j, $S|_{U_j} - h_1 \in \mathrm{good}_{1,2,3}(h_{1.5} - h_1, h_2 - h_1)$.

The $\mathrm{good}_{1,2,3}(g', g)$ condition on excursions T from 0 that exceed g are defined below in terms of their $[0 \uparrow g' \uparrow g]$-decomposition.

The first condition $T \in \mathrm{good}_1(g', g)$ is that T has only one $[g' \uparrow g]$-excursion. That is, T only makes a single excursion away from g' that reaches g. If this condition holds then we define a chain link (recall Section 2.5) as the pair of left and right $[g' \uparrow g]$-end intervals $J_j^- = [\theta^-, \tau^-]$ and $J_j^+ = [\theta^+, \tau^+]$ of T. Notice that $S|_{U_j} - h_1 \in \mathrm{good}_1(h_{1.5} - h_1, h_2 - h_1)$ implies $\tau^- \sim \theta^+$, while $\theta^- \sim \tau^+$ always holds. See Figure 4.4.

If $S \in \mathrm{Good}_1(h_1, h_2)$, then the corresponding collection of chain links $\{(J_j^-, J_j^+)\}_{i=1}^k$ forms a chain of degree k around 0. We call this the (h_1, h_2)-chain associated to S.

Next, we say that $T \in \mathrm{good}_2(g', g)$ if $a_i g'^{-2} \in [L^{-1}, L]$ for $i = 1, 2$, where a_1, a_2 are the lengths of the $[0 \uparrow g' \uparrow g]$-end intervals of T. This condition controls the diameters of the chain link associated to T.

We say that $T \in \mathrm{good}_3(g', g)$ if $\mathbb{e}_{Z_1}$ and $\mathbb{e}_{Z_2}$ are $(L, 1/3)$-Hölder continuous on $[0, 1]$. Here

$$\mathbb{e}_{Z_i}(t) := a_i^{-1/2} Z_i(a_i t) \tag{4.6}$$

are the Brownian rescalings of the $[0 \uparrow g' \uparrow g]$-end intervals of T. This condition gives control over the regularity of the welding on the chain link associated to T.

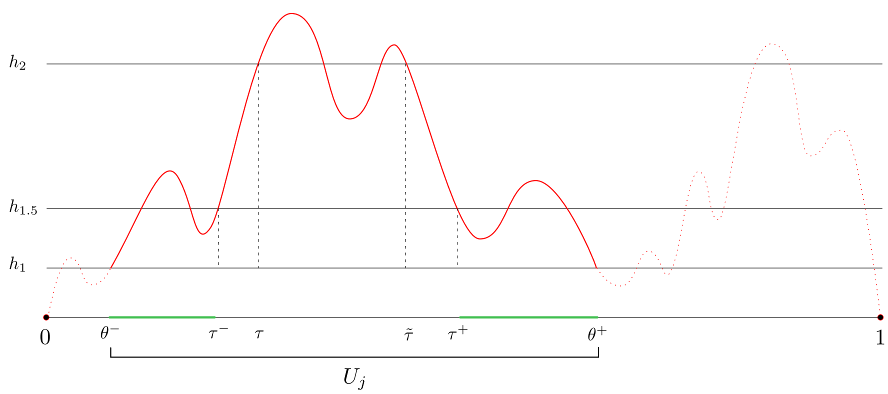


Figure 4.4: Definition of the $\text{good}_1(h_{1.5} - h_1, h_2 - h_1)$ condition. We have highlighted the excursion S over its $j-$th excursion interval U_j. τ and $\tilde{\tau}$ are the first and last hitting times in U_j of height h_2. τ^- is the last hitting time of g' in U_j before hitting h_2, and τ^+ is the first hitting time of g' after $\tilde{\tau}$. We say that $\text{good}_1(h_{1.5} - h_1, h_2 - h_1)$ holds for $S|_{U_j} - h_1$ if the portion of the excursion between τ and $\tilde{\tau}$ does not dip below height $h_{1.5} - h_1$, so that τ^- and τ^+ are identified via $\sim$. If this holds, then the pair $([\theta^-, \tau^-], [\tau^+, \theta^+])$ is a chain link as defined in Section 2.5.

The remaining conditions depends on all the excursion intervals $(S|_{U_j})_{j=1}^k$ at a given scale, simultaneously.

We say that $\mathrm{Good}_4(h_1, h_2)$ holds for S if $n - w_1 - \cdots - w_k \leq L h_1^2$, where $w_1, \ldots, w_k$ are the lengths of the $[h_1 \uparrow h_2]$-excursion intervals, and $\mathrm{Good}_5(h_1, h_2)$ holds if the number k of $[h_1 \uparrow h_2]$-excursion intervals of S is less than L. Finally we say that $\mathrm{Good}_{\tilde{5}}(h_1, h_2)$ holds if $\mathrm{Good}_5(h_1, h_2)$ holds and there is at least one $[h_1 \uparrow h_2]$-excursion interval.

This gives a bound on the degree on the (h_1, h_2)-chain and also the sum (and hence maximum) of the gaps between the chain links.

If all of the conditions are satisfied, we say that $S \in \mathrm{Good}(h_1, h_2)$. Summarizing the discussion above:

Definition 4.2.3. Fix $L, \Lambda > 1$ and $0 < h_1 < h_2$ integer, with $h_2 - h_1 \geq \Lambda$. Let S be an excursion in $\mathbf{E}_n(0)$ and let U_j be the $[0 \uparrow h_1 \uparrow h_2]$-excursion intervals of S. We say that S belongs to $\mathrm{Good}(h_1, h_2)$ if $S|_{U_j} - h_1 \in \mathrm{good}_i(h_{1.5} - h_1, h_2 - h_1)$ for each $1 \leq j \leq k$ and $1 \leq i \leq 3$, and if $\mathrm{Good}_4(h_1, h_2)$ and $\mathrm{Good}_{\tilde{5}}(h_1, h_2)$ holds.

Proposition 4.2.4. *Suppose $h_2 \geq 2h_1$. If an excursion S belongs to $\mathrm{Good}(h_1, h_2)$, then the curve family $\Gamma(\mathcal{C})$ of the (h_1, h_2)-chain $\mathcal{C}$ associated to S satisfies*

$$\operatorname{mod}(\Gamma(\mathcal{C})) \geq \delta_0$$

where $\delta_0 > 0$ depends only on L and Λ.

Proof. We would like to apply Theorem 2.5.1 and need to verify Conditions 1-3.

First, Good_2 implies $|J_j^+| \asymp_{L^2} |J_{j+1}^-|$ and Good_4 implies $|\tau_{j+1}^- - \theta_j^+| \lesssim_{L,\Lambda} |J_j^+|$ so that (2.12) and therefore Condition 1 holds.

Second, Condition 2 follows from Good_3 and (the proof of) Lemma 2.4.6.

And third, the existence of the chain itself and the boundedness of the degree, Condition 3, is implied by Good_1 and $\mathrm{Good}_{\tilde{5}}$. $\square$

4.3 The main estimate and setup: Positive density of good scales

We now formulate the main estimate for the probability that a fixed percentage of scales are good. Fix $\lambda, \Lambda \geq 2$ integer and and consider the sequence of scales $H_0 = 0$ and $H_{l+1} = H_l + \lambda^l$. Let $H_{l+0.5} = H_l + \lfloor \frac{H_{l+1}-H_l}{\Lambda} \rfloor$, for sufficiently large l this will be strictly between H_l and H_{l+1}.

Consider an excursion S of length n, fix $0 < r < 1$ small and consider the ball of radius r centered at the root in the rescaled tree metric $d = d_{S(n\cdot)}/n^{1/2}$ (see (1.2)). We wish to show that it can be separated from a circle of fixed radius by $\gtrsim \log \frac{1}{r}$ annuli of modulus $\gtrsim 1$ with probability $1 - O(r^{T_0})$, where any $T_0 > 2$ suffices for our purpose. More precisely, define ρ such that $H_{\rho-1} < r\sqrt{n} \leq H_\rho$ and define $N \geq \rho$ such that $H_{N-1} < r^{1/2}\sqrt{n} \leq H_N$. Notice that

$$\frac{1}{2}\log_\lambda \frac{1}{r} - 2 \leq N - \rho < \frac{1}{2}\log_\lambda \frac{1}{r} + 2. \tag{4.7}$$

Then many of the associated (H_l, H_{l+1})-chains satisfy the Good conditions and therefore the assumption of Proposition 4.2.4:

Proposition 4.3.1. *There are integers $\lambda, \Lambda, L > 1$ and $r_0 > 0$ such that for all n for which $r \leq r_0$, we have*

$$\mathbb{P}\left(\frac{|\{l = \rho, \ldots, N : S \in \mathrm{Good}_{1,2,3,4,\tilde{5}}(H_l, H_{l+1})\}| + 1}{N - \rho} < \frac{1}{2}\right) \leq C_\lambda r^{2.25},$$

where S is a uniformly random excursion in $\mathbf{E}_n(0)$ and the constant C_λ only depends on λ.

Proposition 4.3.1 follows from a large deviations estimate applied to a Markov chain ω_l and a suitable test function V that we will define in the next section. In the remainder of this section, we prove that each individual $[H_l \uparrow H_{l+1}]-$excursion satisfies the $\mathrm{good}_{1,2,3}$ conditions with probability arbitrarily close to 1 if the parameters are chosen appropriately. We begin with a geometric upper bound on the number of large excursions inside a given excursion. It immediately implies that, at a fixed scale, the Good_5 condition holds with high probability, and later also provides us with control over the Markov chain exploration.

Lemma 4.3.2. *Let $g > 1$ and $\lambda > 1$ be integers. Let S be a uniformly random excursion of type $\mathbf{E}_w(0, \max \geq g)$. Let k be the number of $[g \uparrow \lambda g]$-excursions of S. There exists $p_{\lambda,wg^{-2}}, \tilde{p}_{\lambda,wg^{-2}} < 1$ such that*

$$\mathbb{P}(k \geq m) \leq \tilde{p}_{\lambda}^{m-1} p_{\lambda,wg^{-2}}$$

for $m \geq 1$. Moreover, we can choose $p_{\lambda,wg^{-2}}$ and $\tilde{p}_{\lambda}$ in such a way that

1. $p_{\lambda,wg^{-2}} \lesssim \exp\left(-c_0 \frac{(\lambda-1)^2 g^2}{w}\right)$
2. $\tilde{p}_{\lambda,wg^{-2}} \lesssim \exp\left(-c_0 \frac{(\lambda-1)^2 g^2}{w}\right)$
3. $\tilde{p}_{\lambda,wg^{-2}} \to 0$ *uniformly in* wg^{-2} *as* $\lambda \to \infty$.

Here c_0 is a universal constant and the first two statements are primarily useful when wg^{-2} is bounded.

Proof. Let τ^- and τ^+ be the first and last times respectively that S is at level g. Conditioned on τ^-, τ^+, the walk $S|_{[\tau^-,\tau^+]}$ is, up to translation of the domain, a uniform walk of type $\mathbf{W}_T(g \to g, \min \geq 0)$, where $T = \tau^+ - \tau -$. We have $k \geq 1$ if and only if this latter walk reaches level λg. By Lemma A.1.3, this probability is bounded by a quantity $p_{\lambda,wg^{-2}}$ which has the desired properties.

This proves the statement of the lemma for $m = 1$. For the general case, it suffices to prove the bound $\mathbb{P}(k \geq m+1 | k \geq m) \leq \tilde{p}_{\lambda}$ for $m \geq 1$ and use induction. Suppose S is conditioned on $k \geq m$. Let $U \subset [0, T]$ be the mth excursion interval, and let τ^- be the first time that S hits λg in U. Let τ^+ be the last time in $[0, T]$ that S hits λg. Conditioned on τ^-, τ^+, the walk $S|_{\tau^-,\tau^+}$ is (up to translation of the domain) a uniform walk of type $\mathbf{W}_T(\lambda g \to \lambda g, \min \geq 0)$ where $T = \tau^+ - \tau^-$. We have $k \geq m+1$ if and only if this latter walk hits level g. Thus item 2) follows from Lemma A.1.3, and item 3) follows from Lemma A.1.2. □

Now we are ready to estimate the probability of the good$_{1,2,3}-$ conditions of a single excursion at a fixed level. Let $g, L, \Lambda > 0$ be integers and let $g' = \lfloor g/\Lambda \rfloor$.

Proposition 4.3.3. *For any $W_0 > 0, \epsilon > 0$, there exists $\Lambda_0 > 0$ such that the probability that a uniformly random excursion from 0 of length $w > W_0 g^2$ satisfies the good conditions* $\mathrm{good}_{1,2,3}(g', g)$ *is bounded below by $1 - \epsilon$ when $\Lambda \geq \Lambda_0$ and $L \geq L_0(\Lambda)$.*

Proof. Fix $\epsilon > 0$ and note that Lemma 4.3.2 implies $\mathbb{P}(S \notin \mathrm{good}_1(g', g)) \leq \epsilon$ for sufficiently large Λ. Next, recall the condition $\mathrm{good}_2(g', g)$, which says that $a_i g'^{-2} \in [L^{-1}, L]$, where a_1, a_2 are the lengths of the $[g' \uparrow g]$-end intervals respectively. First we bound the probability p that $a_1 g'^{-2} \notin [L^{-1}, L]$. Notice that if $S \in \mathbf{E}_w(0, \max \geq g)$, the part of S after its first $[g' \uparrow g]$-end interval may be decomposed uniquely into the concatenation of a walk of type $\mathbf{E}(g', \max \geq g)$ and a walk of type $\mathbf{Z}(g' \downarrow 0)$. So, by Corollary A.0.2 and Proposition A.1.1,

$$
\begin{aligned}
p &= \frac{1}{\mathbf{E}_w(0, \max \geq g)} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} |\mathbf{Z}_{a_1}(0 \uparrow g', \max < g)| \cdot |\mathbf{E}_b(g', \max \geq g)| \cdot |\mathbf{Z}_c(g' \downarrow 0)| \\
&\leq \frac{1}{C_{\mathrm{stir}} w^{-3/2} c_{W_0}} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} |\mathbf{Z}_{a_1}(0 \uparrow g', \max < g)| 2^{-a_1} \\
&\quad \cdot |\mathbf{E}_b(0, \max \geq (\Lambda - 1) g')| 2^{-b} \cdot |\mathbf{Z}_c(g' \downarrow 0)| 2^{-c}.
\end{aligned}
$$

Using the estimates from Proposition A.1.1, Lemma A.2.2, and Corollary A.0.2, we get

$$
p \lesssim_{W_0} w^{3/2} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1}, L]}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}}
$$

for some universal constant c_0. Now if $a_1 + b + c = w$ then either $a_1 \geq w/3$, or $b \geq w/3$, or $c \geq w/3$, so the sum above can be bounded by splitting the region of summation over those three regions: we have

$$
p \lesssim w^{3/2} (I_a + I_b + I_c) \tag{4.8}
$$

where, for fixed $\epsilon > 0$ and sufficiently large L,

$$\begin{aligned}
I_a &= \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ a_1 \geq w/3}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq \sup_{\substack{a_1 \geq w/3 \\ a_1 g'^{-2} \notin [L^{-1},L]}} a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ a_1 \geq w/3}} g' \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \epsilon \sum_{b,c \leq w} g' \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \epsilon \cdot \int_1^\infty x^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2}{x}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{3x}} dx \\
&\leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0.
\end{aligned}$$

Similarly, for fixed $\epsilon > 0$ and fixed $\Lambda > 1$, and sufficiently large L ,

$$\begin{aligned}
I_b &= \sum_{\substack{a_1+b+c=w \\ a_1 g'^{-2} \notin [L^{-1},L] \\ b \geq w/3}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq \sup_{b \geq w/3} b^{-3/2} e^{-c_0 \frac{(\Lambda-1)^2 g'^2}{b}} \sum_{\substack{a_1, c \leq w \\ a_1 g'^{-2} \notin [L^{-1},L]}} g' a_1^{-3/2} e^{-\frac{g'^2}{3a_1}} e^{-c_0 \frac{a_1}{g^2}} \cdot g' c^{-3/2} e^{-\frac{g'^2}{3c}} \\
&\leq (w/3)^{-3/2} \cdot \int_{[1,\infty)\setminus[L^{-1},L]} x^{-3/2} e^{-\frac{1}{2x}} e^{-c_0 x \cdot \frac{g'^2}{g^2}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{2x}} dx \\
&\leq (w/3)^{-3/2} \cdot \int_{[1,\infty)\setminus[L^{-1},L]} x^{-3/2} e^{-\frac{1}{2x}} e^{-c_0 x \cdot \frac{1}{2\Lambda}} dx \cdot \int_1^\infty x^{-3/2} e^{-\frac{1}{2x}} dx \\
&\leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0.
\end{aligned}$$

A similar argument gives $I_c \leq (w/3)^{-3/2} \cdot \epsilon \cdot C_0$. Using these estimates in (4.8) gives, for fixed $\epsilon > 0$ and fixed $\Lambda > 1$, and sufficiently large L, $p \leq \epsilon$. By the union bound, the probability that $\mathrm{good}_2(g', g)$ does not hold is bounded by $2p \leq 2\epsilon$.

Finally, we have from Lemma A.0.4 that $\mathbb{P}(S \in \mathrm{good}_3 | S \in \mathrm{good}_2) \geq 1 - \epsilon$ for sufficiently large L. Hence $\mathbb{P}(S \in \mathrm{good}_2 \cap \mathrm{good}_3) \geq (1 - C\epsilon) \cdot (1 - \epsilon)$. By the union bound, we get (for fixed ϵ, for sufficiently large Λ and L),

$$\mathbb{P}(S \in \text{Bad}) \leq \mathbb{P}(S \notin \text{good}_1) + \mathbb{P}(S \notin \text{good}_2 \cap \text{good}_3) \lesssim \epsilon$$

and the proposition follows. $\square$

4.3.1 The Markov chain exploration

The key observation is that a uniformly random $S \in \mathbf{E}_n(0)$ may be explored via a Markov chain on a state space Ω consisting of finite tuples of *quotient excursions*. These are equivalence classes of walks defined via the following equivalence relation on excursions $\mathbf{E}_w(H_l, \max \geq H_{l+1})$: Declare two such excursions S, S' equivalent if they have the same $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals and they are equal on the complement of these excursion intervals. In particular, if S, S' do not reach height H_{l+2} then they are equivalent if and only if they are equal. Denote the equivalence class of S by $[S]$.

Recall the excursion decomposition of Section 4.2.2 and in particular Definition 4.2.1. Equivalence classes have well defined $[H_l \uparrow H_{l+1} \uparrow H_{l+2}]$-ends and -bridges, and well defined $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals. In particular, the conditions $\text{Good}_{1,2,3}(H_l, H_{l+1})$ are well defined on quotient excursions.

If $S \in \mathbf{E}_n(0)$ and if $U_1, \ldots, U_k$ are the $[H_l \uparrow H_{l+1}]$-excursion-intervals of S, then set

$$\omega_l := ([S|_{U_1}], \ldots, [S|_{U_k}])$$

so that $(\omega_l)_{l \geq 1}$ is a Markov chain. To get ω_{l+1} from ω_l,

- Let $V_1, \ldots, V_m$ be the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals of ω_l (this is the collection of $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals over the k quotient excursions in ω_l).

- Independently sample uniformly random excursions in $\mathbf{E}_{|V_j|}(H_{l+1}, \max \geq H_{l+2})$.

- Take the equivalence classes of each of these random excursions.

In particular, this shows that the distribution of ω_{l+1} given ω_l is entirely determined by the lengths of the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals of ω_l. The transition probabilities of this Markov chain can therefore be deduced from (4.5).

We will use the notation $\mathbf{b} = \mathbf{b}(\omega_l)$ for the vector of the lengths of *all* the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals, and denote $k_l = k(\omega_l)$ the total number of these intervals. Note that, with this indexing, ω_l consists of k_{l-1} quotient excursions. We will also write

$$g_l = g(\omega_l) = H_{l+2} - H_{l+1}$$

and

$$\mathrm{Gap}(\omega_l) = n - \sum \beta_i$$

where the sum is over the components β_i of $\mathbf{b}(\omega_l)$. Note that $\mathrm{Gap}(\omega_{l-1})$ can also be determined from ω_l, because ω_l contains the information about the $[H_l \uparrow H_{l+1}]$-decomposition. Now we define the test function for the large deviations estimate. Define $V : \Omega \to \mathbb{R}^+$ by

$$V(\omega_l) = s^{\mathrm{Gap}(\omega_l) g_l^{-2}} \prod_{i=1}^{k_l} q^{\mathbb{1}(\beta_i g_l^{-2} \le W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2} \prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j} - H_l \notin \mathrm{good}_{1,2,3}(H_{l+0.5} - H_l, H_{l+1} - H_l))}. \tag{4.9}$$

For the rest of the paper, we will abbreviate this last term to $q^{\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}$. See the end of Section 4.1 for some heuristic remarks about the function V.

It will be useful to write the test function in the form

$$V(\omega_l) = s^{\mathrm{Gap}(\omega_{l-1}) g_l^{-2}} \prod_{i=1}^{k_{l-1}} v_{g_{l-1} \uparrow (\lambda+1) g_{l-1}} (S|_{U_i} - H_l)^{1/2} q^{\mathbb{1}(S|_{U_i} \notin \mathrm{good}_{1,2,3})}. \tag{4.10}$$

where $v_{h_1 \uparrow h_2} : \mathbf{E}(0, \max \ge h_1) \to \mathbb{R}^+$ is defined by

$$v_{h_1 \uparrow h_2}(S) = s^{2(a_1 + \cdots + a_{k+1}) g^{-2}} \prod_{j=k+2}^{2k+1} (2 + (a_j g^{-2})^{1/4}) q^{2\mathbb{1}(a_j g^{-2} \le W_0)}. \tag{4.11}$$

Here $a_1, \ldots, a_{2k+1}$ is the vector of lengths in the $[0 \uparrow h_1 \uparrow h_2]$-decomposition of S, and $g = h_2 - h_1$. Note that $v_{h_1 \uparrow h_2}$ is well defined on the quotient space of $\mathbf{E}(0, \max \ge h_1)$ described at the beginning of this section.

We need to show that V satisfies the assumptions of Theorem B.0.3. The proof of the following crucial Lemma will occupy the next section.

Lemma 4.3.4. *Set $q = \lambda^{20}$. For sufficiently large $\lambda > 1$ and sufficiently small $s > 1$, the following holds. For sufficiently large $L, \Lambda > 1$, sufficiently small $W_0 > 0$, we have*

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} \le 1 \text{ for all } \omega \in \Omega. \tag{4.12}$$

If $S \notin \mathrm{Good}_{1,2,3}(H_l, H_{l+1})$ or $S \notin \mathrm{Good}_{4,5}(H_{l+1}, H_{l+2})$,

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} < \lambda^{-20}. \tag{4.13}$$

Finally, for $0 < r < 1$,

$$\mathbb{E}V(\omega_\rho) \lesssim_\lambda r^{-1/4}. \tag{4.14}$$

where ρ satisfies $H_{\rho-1} < r\sqrt{n} \le H_\rho$.

Proof. Recall that ω_{l+1} is generated by the k_l independent excursions $T_1, \ldots, T_{k_l}$, where T_i is uniformly randomly chosen from $\mathbf{E}_{\beta_i}(H_{l+1}, \max \ge H_{l+2})$. Using (4.9) for the denominator and (4.10) for the numerator, we can write

$$\begin{aligned}\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} &= \frac{s^{\mathrm{Gap}(\omega_l)g_{l+1}^{-2}} \prod_{i=1}^{k_l} \mathbb{E}\left[v_{g_l\uparrow(\lambda+1)g_l}(T_i)^{1/2} q^{\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}\right]}{s^{\mathrm{Gap}(\omega_l)g_l^{-2}} \prod_{i=1}^{k_l} q^{\mathbb{1}(\beta_i g^{-2} \le W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2} \prod_{j=1}^{k_{l-1}} q^{\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}} \\ &= s^{-\mathrm{Gap}(\omega_l)(1-\lambda^{-2})g_l^{-2}} \left(\prod_{i=1}^{k_l} \frac{\mathbb{E}\left[v_{g_l\uparrow(\lambda+1)g_l}(T_i)^{1/2} q^{\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}\right]}{q^{\mathbb{1}(\beta_i g_l^{-2} \le W_0)} (2 + (\beta_i g_l^{-2})^{1/4})^{1/2}} \right) \prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})},\end{aligned}$$

where the expectations are with respect to independent, uniformly random $T_i \in \mathbf{E}_{\beta_i}(0, \max \ge g_l)$.

By the Cauchy-Schwarz inequality, each term in the middle product is bounded above by

$$\left(\frac{\mathbb{E}v_{g_l\uparrow(\lambda+1)g_l}(T_i)}{2 + (\beta_i g_l^{-2})^{1/4}} \right)^{1/2} \cdot \left(\frac{\mathbb{E}q^{2\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}}{q^{2\mathbb{1}(\beta_i g_l^{-2} \le W_0)}} \right)^{1/2}. \tag{4.15}$$

Set $\lambda = \lambda_0 + 1$ where λ_0 is the constant of Lemma 4.3.5, and set $q = \lambda^{20}$, and W_0 to be the constant of Lemma 4.3.5. Choose $s < s_\lambda$ small enough that

$$s^{2g_{\rho-1}^{-2}} \cdot \exp\left(-c_0 \frac{1}{H_\rho^2}\right) \leq 1. \tag{4.16}$$

This last condition on s will only be used further below in the proof of (4.14).

For these parameters, we have from Lemma 4.3.5 that the first term in (4.15) is bounded by $\left(\frac{7}{8}\right)^{1/2}$.

Now we turn to the second term of (4.15). By Proposition 4.3.3 with $\epsilon = 0.01q^{-2}$ there is a $\Lambda > 0$ such that for $L \geq L_0(\Lambda)$ we have

$$\frac{\mathbb{E}q^{2\mathbb{1}(T_i \notin \mathrm{good}_{1,2,3})}}{q^{2\mathbb{1}(\beta_i g_l^{-2} \leq W_0)}} \leq 1 + q^2 \mathbb{P}(T_i \notin \mathrm{good}_{1,2,3} | \beta_i g_l^{-2} > W_0) \leq 1.01.$$

It follows that the product (4.15) is bounded by 0.95. Thus we get

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} \leq s^{-\mathrm{Gap}(\omega)(1-\lambda^{-2})g_l^{-2}} 0.95^{k_l} \prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}. \tag{4.17}$$

This immediately implies (4.12). Now suppose $S|_{U_j} \notin \mathrm{good}_{1,2,3}$ for some j, or $S \notin \mathrm{Good}_{4,5}(H_{l+1}, H_{l+2})$. In the latter case this implies that $k_l \geq L$ or $\mathrm{Gap}(\omega_l) g_l^{-2} \geq L$. Then

$$\frac{\mathbb{E}[V(\omega_{l+1})|\omega_l = \omega]}{V(\omega)} \leq \max\left(s^{-L(1-\lambda^{-2})}, 0.95^L, \frac{1}{q}\right)$$

and this last expression can be made to be smaller than λ^{-20} by taking L large. This proves (4.13).

To prove the last inequality (4.14), we will show that

$$\mathbb{E}[V(\omega_\rho)] \leq \mathbb{E}[v_{H_\rho \uparrow H_{\rho+1}}(S)]^{1/2} \lesssim (2 + (n/g_{\rho-1}^2)^{1/4})^{1/2} \leq C_\lambda r^{-1/4}, \tag{4.18}$$

where the last inequality is clear from the definition of ρ.

For the first inequality, let $V'(\omega_l) = V(\omega_l) \prod_{j=1}^{k_{l-1}} q^{-\mathbb{1}(S|_{U_j} \notin \mathrm{good}_{1,2,3})}$ and notice that $V'(\omega_l)$ is $\mathbf{b}(\omega_l)$-measurable. Recall that $\mathbf{b}(\omega_l)$ is the vector of lengths of the $[H_{l+1} \uparrow H_{l+2}]$-excursion intervals. Thus

$$\mathbb{E}V(\omega_\rho) = \mathbb{E}\left[\mathbb{E}\left[V(\omega_\rho)|\mathbf{b}(\omega_{\rho-1})\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{V(\omega_\rho)}{V'(\omega_{\rho-1})}\middle|\mathbf{b}(\omega_{\rho-1})\right] V'(\omega_{\rho-1})\right] \leq \mathbb{E}[V'(\omega_{\rho-1})],$$

where the inequality is from (4.17), using the fact that conditioning on $\omega_{\rho-1}$ is the same as conditioning on $\mathbf{b}(\omega_{\rho-1})$, and the fact that $s > 1$. This last expectation is, by (4.9), equal to

$$\begin{aligned}\mathbb{E}[V'(\omega_{\rho-1})] &= \mathbb{E}\left[s^{(a_1+\cdots+a_{k+1})g_{\rho-1}^{-2}} \prod_{i=k+2}^{2k+1} q^{\mathbb{1}(a_i g_{\rho-1}^{-2} \leq W_0)}(2+(a_i g_{\rho-1}^{-2})^{1/4})^{1/2}\right] \\ &= \mathbb{E} v_{H_\rho \uparrow H_{\rho+1}}(S)^{1/2} \leq \mathbb{E}[v_{H_\rho \uparrow H_{\rho+1}}(S)]^{1/2},\end{aligned}$$

where S is a uniformly random element of $\mathbf{E}_n(0)$ and the $a_1, \ldots, a_{2k+1}$ are the lengths in the $[0 \uparrow H_\rho \uparrow H_{\rho+1}]$-decomposition.

For the second inequality of (4.18), apply Lemma 4.3.5 with our choices of q and W_0, and $\mu = \frac{H_{\rho+1}}{H_\rho} - 1 \in [\lambda_0, \lambda_0 + 1]$ to obtain

$$\mathbb{E}[v_{H_\rho \uparrow H_{\rho+1}}(S) | \max S \geq H_\rho] \leq \frac{7}{8}(2 + (n g_{\rho-1}^{-2})^{1/4}).$$

On the other hand, we have by Proposition A.1.1 that $\mathbb{P}(\max S < H_\rho) \lesssim \exp(-c_0 \frac{n}{H_\rho^2})$, while $\mathbb{E}[v_{H_\rho \uparrow H_{\rho+1}}(S) | \max S < H_\rho] = s^{2n g_{\rho-1}^{-2}}$. Thus by our choice of s, (4.16), we are done. $\square$

Now that we have proved that our test function V satisfies the hypotheses of Theorem B.0.3, we can apply the theorem and prove that most of the scales are good.

Proof of Proposition 4.3.1. Choose the constants $L, \Lambda, s, \lambda, W_0$ so that the conclusion of Lemma 4.3.4 holds. Applying Theorem B.0.3 (with $\epsilon = \frac{1}{4}$) to the Markov chain $(\omega_l)_{l \geq \rho}$ and the test function $u(\omega_l) = V(\omega_l)$ yields

$$\begin{aligned}&\mathbb{P}\left(\frac{|\{l = \rho, \ldots, N : S \in \mathrm{Good}_{1,2,3}(H_l, H_{l+1}) \text{ and } S \in \mathrm{Good}_{4,5}(H_{l+1}, H_{l+2})\}|}{N - \rho} < \frac{3}{4}\right) \\ &\lesssim r^{-1/4}(\lambda^{-5})^{(N-\rho+1)} \leq r^{-1/4}(\lambda^{-1} r^{-1/2})^{-5} \leq \lambda^5 r^{2.25}.\end{aligned}$$

Here we have used (4.7).

To improve the Good_5 above into $\mathrm{Good}_{\tilde{5}}$, we use the union bound together with the following observation. If $k_l = 0$ for some $l \leq N$ then $\max S \leq H_{N+2}$. Thus if $\max S > H_{N+2}$ then $\mathrm{Good}_5(H_l, H_{l+1})$ implies $\mathrm{Good}_{\tilde{5}}(H_l, H_{l+1})$. Now $H_{N+2} < \lambda^4 r^{1/2} \sqrt{n}$, and by Proposition

A.1.1, $\mathbb{P}_{\mathbf{E}_n(0)}(\max S \leq \lambda^4 r^{1/2}\sqrt{n}) \leq \frac{3}{2}\exp\left(-c_0\frac{1}{\lambda^8 r}\right)$ which is bounded by $r^{2.25}$ for r sufficiently small.

We have shown that the proportion of scales $l = \rho, \ldots, N$ that do not satisfy $\mathrm{Good}_{1,2,3}(H_l, H_{l+1})$ is bounded by $1/4$, and likewise the proportion of scales that do not satisfy $\mathrm{Good}_{4,\tilde{5}}(H_l, H_{l+1})$ is bounded by $1/4 + 1/(N-\rho)$. The statement of the proposition follows. □

4.3.2 Proof of bound for v

Recall the definition of v in (4.11).

Lemma 4.3.5. *There exists μ_0 such that for g, h integer with $\mu := h/g - 1 \geq \mu_0$, there exists $W_0 > 0$ and $s = s(\mu)$ such that for all $1 < s < s(\mu)$ and $q \leq (\mu+1)^{20}$, we have*

$$\frac{\mathbb{E}v_{g\uparrow h}(S)}{2 + (wg^{-2})^{1/4}} \leq \frac{7}{8}$$

for all $w > 0$ even, and $g \geq 1$ integer. Here the expectation is with respect to S being a uniformly random element of $\mathbf{E}_w(0, \max \geq g)$.

Proof. First we assume that $wg^{-2} \leq 1$. In this case, it is likely that $k = 0$, and it suffices to use the bounds $(2 + (wg^{-2})^{1/4}) \leq 3$ and $(a_1 + \cdots + a_{k+1})g^{-2} \leq 1$ and $q^{\mathbb{1}(ag^{-2} \leq W_0)} \leq q$. We get

$$\begin{aligned}
\frac{\mathbb{E}(v_{g\uparrow(\mu+1)g}(\mathbf{a}_i))}{2 + (wg^{-2})^{1/4}} &\leq \frac{s^{2\mu^{-2}}\mathbb{E}\left(3^k q^{2k}\right)}{2} \\
&\leq \frac{s^{2\mu^{-2}}}{2}\left(1 + p\sum_{k=1}^{\infty}\tilde{p}^{k-1}3^k q^{2k}\right) \\
&= \frac{s^{2\mu^{-2}}}{2}\left(1 + p\frac{3q^2}{1 - 3\tilde{p}q^2}\right)
\end{aligned}$$

In the second inequality we have used Lemma 4.3.2, where p and $\tilde{p}$ are the probabilities in the conclusion of that lemma. Recall that they converge to 0 exponentially in $(\mu-1)^2$ as $\mu \to \infty$ (when wg^{-2} is bounded). Thus for any $s > 1$, if q is polynomial in μ, this expression is bounded by $\frac{7}{8}$ as long as μ is sufficiently large.

Now we turn to the case $wg^{-2} > 1$. By (4.5), we have

$$\frac{\mathbb{E}v_{g\uparrow(\mu+1)g}(S)}{2+(wg^{-2})^{1/4}} = \frac{|\mathbf{E}_w(0,\max \geq g, \max < (\mu+1)g)|}{|\mathbf{E}_w(0,\max \geq g)|}\frac{s^{2wg^{-2}\mu^{-2}}}{2+(wg^{-2})^{1/4}} + I \tag{4.19}$$

where

$$\begin{aligned} I =& \frac{1}{2+(wg^{-2})^{1/4}}\frac{1}{|\mathbf{E}_w(0,\max \geq g)|}\sum_{k=1}^{\infty}\sum_{a_1+\cdots+a_{2k+1}=w} \\ & s^{2a_1g^{-2}\mu^{-2}}|\mathbf{Z}_{a_1}(0\uparrow g, \max < (\mu+1)g)| \cdot s^{2a_2g^{-2}\mu^{-2}}|\mathbf{Z}_{a_2}(g\downarrow 0, \max < (\mu+1)g)| \\ & \times \prod_{i=3}^{k+1} s^{2a_ig^{-2}\mu^{-2}}|\mathbf{W}^{\uparrow}_{a_i}(g\to g, 0 \leq \min < \max < (\mu+1)g)| \\ & \times \prod_{i=k+2}^{2k+1} q^{2\mathbb{1}(a_i\mu^{-2}g^{-2}\leq W_0)}|\mathbf{E}_{a_i}(g, \max \geq (\mu+1)g)| \cdot (2+(a_i\mu^{-2}g^{-2}\mu^{-2})^{1/4}). \end{aligned}$$

Multiplying each term in the sum by

$$1 = \mu^{-k}\frac{1}{g2^{-w}}2^{-a_1}2^{-a_2}\prod_{i=3}^{k+1} g^{-1}2^{-a_i}\prod_{i=k+1}^{2k+1} 2^{-a_i}\mu g,$$

we can write $I = \sum_{k=1}^{\infty} A_k$ where

$$A_k = \frac{\mu^{-k}\sum_{a_1+\cdots+a_{2k+1}=w} F_{\mathbf{Z}}(a_1)F_{\mathbf{Z}}(a_2)\prod_{i=3}^{k+1} F_{\mathbf{B}}(a_i)\prod_{i=k+2}^{2k+1} F_{\mathbf{E}}(a_i)}{(2+(wg^{-2})^{1/4})g|\mathbf{E}_w(0,\max \geq g)|2^{-w}}$$

where

$$\begin{aligned} F_{\mathbf{Z}}(a) &= s^{2ag^{-2}\mu^{-2}}|\mathbf{Z}_a(0\uparrow g, \max < (\mu+1)g)| \cdot 2^{-a} \\ F_{\mathbf{B}}(a) &= g^{-1}s^{2ag^{-2}\mu^{-2}}|\mathbf{W}^{\uparrow}_a(0\to 0, -g \leq \min < \max < \mu g)| \cdot 2^{-a} \\ F_{\mathbf{E}}(a) &= \mu g q^{2\mathbb{1}(a\mu^{-2}g^{-2}\leq W_0)}|\mathbf{E}_a(0,\max \geq \mu g)| \cdot 2^{-a}(2+(ag^{-2}\mu^{-2})^{1/4}). \end{aligned} \tag{4.20}$$

We bound A_k using the following observation, valid for any positive functions $F_1, \ldots, F_{2k+1}$:

$$\begin{aligned} \frac{1}{Z}\sum_{a_1+\cdots+a_{2k+1}=w}\prod_{i=1}^{2k+1} F_i(a_i) &\leq \frac{1}{Z}\sum_{j=1}^{2k+1}\sum_{\substack{a_1+\cdots+a_{2k+1}=w \\ a_j\geq w(2k+1)^{-1}}}\prod_{i=1}^{2k+1} F_i(a_i) \\ &\leq \sum_{j=1}^{2k+1}\sup_{w(2k+1)^{-1}\leq a\leq w}\frac{F_j(a)}{Z}\prod_{\substack{i=1 \\ i\neq j}}^{2k+1}\sum_{a=0}^{\infty} F_i(a_i). \end{aligned}$$

Now Lemmas 4.3.6, 4.3.7 and 4.3.8 below show that, given ϵ, if $\mu > 1$ is sufficiently large and $s > 1$ is sufficiently small, then for all $q > 1$ there exists L large and W_0 small such that

$$\begin{aligned} A_k &\le \mu^{-k} \cdot [2 \cdot C_{\mathbf{Z}}(2k+1)^{3/2}\Sigma_{\mathbf{Z}}\Sigma_{\mathbf{W}}^{k-1}\Sigma_{\mathbf{E}}^{k} + (k-1) \cdot C_{\mathbf{W}}(2k+1)^{3/2}\Sigma_{\mathbf{Z}}^{2}\Sigma_{\mathbf{W}}^{k-2}\Sigma_{\mathbf{E}}^{k} \\ &\qquad + k \cdot \mu(2k+1)^{3/2}\epsilon\Sigma_{\mathbf{Z}}^{2}\Sigma_{\mathbf{W}}^{k-1}\Sigma_{\mathbf{E}}^{k-1}] \\ &\le C^{2k+1}\mu^{-k}(2k+1)^{3/2}(k+1) + C^{2k}\mu^{-k+1}(2k+1)^{5/2}k\epsilon, \end{aligned} \tag{4.21}$$

where in the last line we have absorbed all the constants $C_{\dots}, \Sigma_{\dots}$ into a single constant C.

Therefore

$$I \le \sum_{k=1}^{\infty} C^{2k+1}\mu^{-k}(2k+1)^{3/2} + C^{2k}\mu^{-k+1}(2k+1)^{5/2}k\epsilon.$$

Taking ϵ small and μ large gives $I \le 1/8$. Turning back to the rest of (4.19), we have

$$\begin{aligned} \frac{|\mathbf{E}_w(0, \max \ge g, \max < (\mu+1)g)|}{|\mathbf{E}_w(0, \max \ge g)|}\frac{s^{2wg^{-2}\mu^{-2}}}{2} &\le \mathbb{P}_{\mathbf{E}_w(0)}(\max S < (\mu+1)g)\frac{s^{2wg^{-2}\mu^{-2}}}{2} \\ &\le \frac{3}{2}e^{-c_0 w(\mu+1)^{-2}g^{-2}}\frac{s^{2wg^{-2}\mu^{-2}}}{2}. \end{aligned}$$

Here we used Proposition A.1.1 for the second inequality. For sufficiently small $s > 1$, this is bounded above by $\frac{3}{4}$.

Together with our bound $I \le 1/8$, this proves the desired estimate when $wg^{-2} > 1$. $\square$

6 inequalities

In this subsection we prove the inequalities needed in the proof of Lemma 4.3.5. For the definitions of $F_{\mathbf{Z}}, F_{\mathbf{E}}$ and $F_{\mathbf{B}}$, see (4.20), and let

$$Z = g|\mathbf{E}_w(0, \max \ge g)|2^{-w} \cdot (2 + (wg^{-2})^{1/4}).$$

Lemma 4.3.6. *There exists $\Sigma_{\mathbf{E}} > 0$ such that for $\mu \ge 2$ sufficiently large, all $q > 1$, and sufficiently small W_0 we have*

$$\sum_{a=1}^{\infty} F_{\mathbf{E}}(a) \le \Sigma_{\mathbf{E}} \tag{4.22}$$

uniformly in g. Furthermore, for every $\epsilon > 0$,

$$\sup_{w \geq a \geq w(2k+1)^{-1}} \frac{F_{\mathbf{E}}(a)}{Z} \leq \mu(2k+1)^{3/2}\epsilon \tag{4.23}$$

if μ is large and W_0 small enough.

Proof. (4.22) is equivalent to: For all $\mu \geq 2$ sufficiently large, for all $q > 1$, for sufficiently small W_0,

$$\sum_{a=1}^{\infty} \mu g q^{\mathbb{1}(ag^{-2} \leq W_0 \mu^2)} |\mathbf{E}_a(0, \max \geq \mu g| \cdot 2^{-a} \cdot (2 + (ag^{-2}\mu^{-2})^{1/4}) \leq \Sigma_{\mathbf{E}}.$$

The left hand side only depends on μg, so we may replace μg with g. We will split the sum into two parts: $ag^{-2} \leq W_0$ and $ag^{-2} > W_0$. We have, by Stirling's approximation and Proposition A.1.1,

$$\begin{aligned}
&\sum_{a > g^2 W_0}^{\infty} g q^{\mathbb{1}(ag^{-2} < W_0)} |\mathbf{E}_a(0, \max \geq g)| \cdot 2^{-a} \cdot (2 + (ag^{-2})^{1/4}) \\
&\lesssim \sum_{a=1}^{\infty} g \exp\left(-c_0 \frac{g^2}{a}\right) a^{-3/2}(2 + (ag^{-2})^{1/4}) = \sum_{a=1}^{\infty} \frac{1}{g^2} \exp\left(-c_0 \frac{g^2}{a}\right) (ag^{-2})^{-3/2}(2 + (ag^{-2})^{1/4}) \\
&\lesssim \int_0^{\infty} x^{-3/2} \exp\left(-c_0 \frac{1}{x}\right) (2 + x^{1/4}) dx.
\end{aligned} \tag{4.24}$$

On the other hand, a similar sequence of computations shows that

$$\sum_{a \leq g^2 W_0} qg |\mathbf{E}_a(0, \max \geq g)| \cdot 2^{-a} \cdot (2 + (ag^{-2})^{1/4}) \lesssim q \int_0^{W_0} x^{-3/2} \exp\left(-c_0 \frac{1}{x}\right) (2 + x^{1/4}) dx.$$

Combining this with (4.24) proves (4.22).

Now we turn to the second inequality of the lemma. Choose $\mu_1 = \mu_1(\epsilon)$ large enough that

$$\sup_{w \geq a \geq w(2k+1)^{-1}} \frac{2 + (ag^{-2}\mu^{-2})^{1/4}}{2 + (wg^{-2})^{1/4}} < \epsilon \tag{4.25}$$

whenever $wg^{-2} > \mu_1^2$ and $\mu > \mu_1$. By Proposition A.1.1 (parts a and c), there exists $\mu_2 > 1$ such that if $\mu > \mu_2$ then

$$\frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq g)} \lesssim \exp\left(-c_0 \mu^2 g^2 / a\right) \leq \epsilon, \text{ whenever } ag^{-2} \leq \mu_1^2. \tag{4.26}$$

Choose $\mu > \max(\mu_1, \mu_2)$. The same proposition also shows that if $0 < W_0 < W_0(\mu, q, \epsilon)$ is sufficiently small, then

$$\frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq g)} \leq \epsilon q^{-1}, \text{ whenever } ag^{-2} < W_0. \tag{4.27}$$

By Stirling's approximation and the monotonicity Lemma A.2.3,

$$\begin{aligned}\frac{|\mathbf{E}_a(0, \max \geq \mu g)|2^{-a}}{|\mathbf{E}_w(0, \max \geq g)|2^{-w}} &= \frac{|\mathbf{E}_a(0)|2^{-a} \cdot \mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{|\mathbf{E}_w(0)|2^{-w} \cdot \mathbb{P}_{\mathbf{E}_w(0)}(\max S \geq g)} \\ &\leq C_0 \left(\frac{a}{w}\right)^{-3/2} \cdot \frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0\to 0)}(\max S \geq g)}. \end{aligned} \tag{4.28}$$

From this we see that to prove (4.23) it suffices to show that

$$\sup_{w \geq a \geq w(2k+1)^{-1}} q^{2\mathbb{1}(a\mu^{-2}g^{-2} \leq W_0)} \frac{\mathbb{P}_{\mathbf{E}_a(0)}(\max S \geq \mu g)}{\mathbb{P}_{\mathbf{E}_a(0\to 0)}(\max S \geq g)} \cdot \frac{2 + (ag^{-2}\mu^{-2})^{1/4}}{2 + (wg^{-2})^{1/4}} \leq \epsilon.$$

The case $W_0 \leq ag^{-2} \leq \mu_1^2$ follows from (4.26), whereas (4.27) takes care of the case $ag^{-2} < W_0$, and the case $ag^{-2} > \mu_1^2$ follows from (4.25) and (4.28). □

Lemma 4.3.7. *There exist constants $\Sigma_{\mathbf{W}}, C_{\mathbf{W}} < \infty$ so that the following holds. For $\mu \geq 2$, there exists $s > 1$ such that*

$$\sum_{a=1}^{\infty} F_{\mathbf{B}}(a) \leq \Sigma_{\mathbf{W}} \tag{4.29}$$

uniformly in g, and if $wg^{-2} \geq 1$, then for all $k > 1$,

$$\sup_{w(2k+1)^{-1} \leq a \leq w} \frac{F_{\mathbf{B}}(a)}{Z} < C_{\mathbf{W}}(2k+1)^{3/2}. \tag{4.30}$$

Proof. Let $p = \mathbb{P}^{\mathbf{W}_a^{\uparrow}(0\to 0)}(-g \leq \min S < \max S < \mu g)$. Then by (A.7),

$$p \leq \mathbb{P}^{\mathbf{W}_a^{\uparrow}(0\to 0)}(-g \leq \min S) \leq 4(ag^{-2})^{-1}.$$

Using this together with Proposition A.1.1c) gives

$$p \leq \min(3/2e^{-c_0\mu^{-2}ag^{-2}}, 4a^{-1}g^2).$$

We have, by Stirling's approximation (Lemma A.4)

$$\begin{aligned}|\mathbf{W}_a^{\uparrow}(0 \to 0, -g \le \min < \max < \mu g)| \cdot 2^{-a} &= |\mathbf{W}_a^{\uparrow}(0 \to 0)| \cdot 2^{-a} p \\ &\le C_{\mathrm{stir}} a^{-1/2} \min(3/2 e^{-c_0 \mu^{-2} a g^{-2}}, 4a^{-1} g^2).\end{aligned}$$

Now choose T large enough and s small enough that $C_0 e^{-c_0 \mu^{-2} x} s^{2\mu^{-2} x} \le 8x^{-1}$ for real numbers $x \ge T$. By making $s > 1$ smaller if necessary, we can assume that $s^{2x\mu^{-2}} \le 2$ for $x \in [0, T]$. Later in the proof we will also need to assume that $T \ge 1$.

By considering the cases $2ag^{-2} \ge T$ and $2ag^{-2} < T$ separately, we get

$$s^{ag^{-2}\mu^{-2}} |\mathbf{W}_a^{\uparrow}(0 \to 0, -g \le \min < \max < \mu g)| \cdot 2^{-a} \le 8 C_{\mathrm{stir}} a^{-3/2} g^2. \tag{4.31}$$

To prove (4.29), we split the sum into two parts and use (4.31) to bound the right sum:

$$\begin{aligned}\sum_{a=1}^{\infty} F_{\mathbf{B}}(a) &\le \sum_{a \le g^2} g^{-1} \cdot s^{ag^{-2}\mu^{-2}} C_{\mathrm{stir}} a^{-1/2} + \sum_{a > g^2} g^{-1} 8 C_{\mathrm{stir}} a^{-3/2} g^2 \\ &\le \sum_{a \le g^2} g^{-1} \cdot 2C_{\mathrm{stir}} \cdot a^{-1/2} + \sum_{a > g^2} g \cdot 8 C_{\mathrm{stir}} a^{-3/2},\end{aligned}$$

and both sums are bounded by a constant $\Sigma_{\mathbf{B}}$ independent of g. This completes the proof of (4.29).

For the other statement (4.30), we have from Proposition A.1.1b) and Stirling's approximation (Lemma A.4) that there is a constant $c > 0$ such that

$$|\mathbf{E}_w(0, \max \ge g)| \cdot 2^{-w} \ge C_{\mathrm{stir}}^{-1} a^{-3/2} c \qquad \text{whenever } wg^{-2} \ge 1. \tag{4.32}$$

Together with (4.31), this immediately implies (4.30). $\square$

Lemma 4.3.8. *There exist constants $\Sigma_{\mathbf{Z}}, C_{\mathbf{Z}} < \infty$ such that the following holds. For $\mu \ge 2$, for sufficiently small $s > 1$*

$$\sum_{a=1}^{\infty} F_{\mathbf{Z}}(a) \le \Sigma_{\mathbf{Z}} \tag{4.33}$$

uniformly in g, and if $wg^{-2} \ge 1$, then for all $k \ge 1$,

$$\sup_{w \ge a \ge w(2k+1)^{-1}} \frac{F_{\mathbf{Z}}(a)}{Z} < C_{\mathbf{Z}}(2k+1)^{3/2}. \tag{4.34}$$

Proof. We have by Lemma A.2.2 and Proposition A.1.1c),

$$\mathbb{P}^{\mathbf{Z}_a(0\uparrow g)}(\max S \leq (\mu+1)g) \leq 3/2 e^{-c_0(\mu+1)^{-2}ag^{-2}} \tag{4.35}$$

for some universal constant $c_0 > 0$. Substituting (A.3) and (4.35) into the sum (4.33) gives

$$\begin{aligned}\sum_{a=1}^{\infty} F_{\mathbf{Z}}(a) &\leq \sum_{a=2}^{\infty} 2C_{\text{stir}} g^{-2}(a/g^2)^{-3/2} \cdot e^{-\frac{g^2}{3a}} e^{-c_0(\mu+1)^{-2}ag^{-2}} s^{2ag^{-2}\mu^{-2}} \\ &\leq 2C_{\text{stir}} \sum_{a=2}^{\infty} g^{-2}(a/g^2)^{-3/2} e^{-\frac{g^2}{3a}} \\ &\lesssim \int_0^{\infty} x^{-3/2} \cdot e^{-1/x} dx =: \Sigma_{\mathbf{Z}}.\end{aligned}$$

where the second last inequality holds as long as $s^{2\mu^{-2}} \leq e^{c_0(\mu+1)^{-2}}$. This proves the first statement of the Lemma.

The second statement follows again from (A.3) and (4.32). $\square$

4.4 Finishing the proof

4.4.1 The measures are tight

In this section, we combine the results of the previous sections to prove a Hölder continuity estimate for the welding solutions ι_n.

To this end, we first note that Brownian scaled excursions $\mathbb{e}_S := n^{-1/2}S(n\cdot)$ are Hölder continuous with high probability: Indeed, the proof of Lemma A.0.4 can be adapted to show that for every Hölder exponent $\delta < 1/2$ there is a constant $C = C_{\epsilon,\delta}$ such that

$$\mathbb{P}(\mathbb{e}_S \text{ is } (C,\delta)\text{-Hölder}) \geq 1-\epsilon$$

independently of n. See Lemma 1.5.1 of [9] for a different proof. Denoting $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$ the associated tree and $p : \mathbb{T} \to \mathcal{T}_{\mathbb{e}}$ the quotient map, it follows that

$$\mathbb{P}(p \text{ is } (2C,\delta)\text{-Hölder}) \geq 1-\epsilon.$$

Fix n and denote A the set of excursions of length $2n$ for which p is $(2C,\delta)$-Hölder. Let $I \subset \mathbb{T}$ be an interval containing 0 and $S \in A$. Then $p(I)$ contains the root and has diameter

$< 2C\mathrm{diam}(I)^{\delta}$. Denote J the ball of radius $r = 2C\mathrm{diam}(I)^{\delta}$ centered at the root. By Proposition 4.3.1 and Proposition 4.2.4, $f(I)$ can be separated from a circle of fixed radius by a family of curves of modulus $M > \delta_0 \log_{\lambda}(1/r)/2$ with probability

$$\mathbb{P} > 1 - Cr^{2.25} = 1 - C'\mathrm{diam}(I)^{2.25\delta}.$$

By Lemma 2.3.2, we have

$$\mathrm{diam}(f(I)) \lesssim \exp(-2\pi M) \leq r^{\pi\delta_0/\log\lambda} = C\mathrm{diam}(I)^{\delta\pi\delta_0/\log\lambda}$$

on this event. By the rotational invariance of the uniform arc pairing lamination, we get the same estimate for all intervals $I \subset \mathbb{T}$, not only those containing 0, if we restrict to the excursions in A. Set

$$\alpha = \delta\pi\delta_0/\log\lambda$$

and consider dyadic intervals $I_{j,k}$ of size 2^{-k}. By the above estimate we can make

$$\sum_{(j,k):k\geq k_0} \mathbb{P}(\mathrm{diam}(f(I)) > C\mathrm{diam}(I)^{\alpha}) < \epsilon$$

by choosing k_0 large, provided that we have chosen δ close enough to $1/2$ that $2.25\delta > 1$. Since every interval $I \subset \mathbb{T}$ can be covered by two adjacent dyadic intervals of lesser size, we have

$$\mathbb{P}(\mathrm{diam}(f(I)) > 2C\mathrm{diam}(I)^{\alpha} \text{ for some } I \text{ with } \mathrm{diam}(I) < 2^{-k_0}) < \epsilon.$$

By the Arzela-Ascoli theorem and Prokhorov's theorem, this implies that the sequence of measures μ_n are tight. Note that the Hölder exponent α is universal and can in principle be computed.

4.4.2 Uniqueness of subsequential limits

We have proved that the law of the conformal welding solutions to the uniform random arc pairing laminations converges along subsequences. All that remains is to show that the limiting random maps do solve the Brownian lamination. The main steps in the proof of this fact are contained in Lemmas 4.4.1, 4.4.5 and Proposition 4.4.4. As a consequence of our proof, we get a Hölder continuous solution to the Brownian lamination. By a result of Jones and Smirnov [36], this means that the welding to the Brownian lamination is unique. It follows that the conformal welding solutions ι_n converge (not just along subsequences). This last part of the argument is given at the end of this section, and completes the proof of Theorem 1.3.2.

We topologize the space of laminations by considering it as a subset of the compact subsets of $\mathbb{T} \times \mathbb{T}$ with the Hausdorff topology.

We first prove a continuity statement about the equivalence relation induced by an excursion. The argument already appears in [9, Proposition 1.3.2].

Lemma 4.4.1. *Let $\mathcal{E} \subset \mathcal{C}([0,1])$ be the set of excursions $e : [0,1] \to \mathbb{R}^+$. Let $e \in \mathcal{E}$ be an excursion for which $\sim_e$ has degree at most 3. If $e_n \in \mathcal{E}$ is a sequence converging to e with respect to the sup norm, then $\sim_{e_n} \to \sim_e$.*

Proof. Let e be an excursion and suppose $e_n \to e$ are excursions converging uniformly to e. Let $d_n, d : \mathbb{T}^2 \to [0,\infty)$ be the pseudometrics corresponding to e_n and e respectively. Note that the map taking an excursion to its pseudometric is continuous and in fact Lipschitz of constant 4. In particular d_n converges uniformly to d. Let $\sim_n = \{(s,t) : d_n(s,t) = 0\} \subset \mathbb{T} \times \mathbb{T}$ and $\sim = \{(s,t) : d(s,t) = 0\}$ be the equivalence relations induced by e_n and e.

Let n_k be a subsequence for which $\sim_{n_k}$ converges in $\mathcal{K}(\mathbb{T} \times \mathbb{T})$, and let $\sim_\infty$ denote the subsequential limit.

The lemma will be proved if we can show that $\sim_\infty = \sim$. First we show that that $\sim_\infty \subset \sim$. If $(s,t) \in \sim_\infty$ this means that for each n_k there exists $(s_{n_k}, t_{n_k}) \in \sim_{n_k}$ such that $(s_{n_k}, t_{n_k}) \to (s,t)$. This means that $d_{n_k}(s_{n_k}, t_{n_k}) = 0$, and so $d(s,t) = 0$ because d_{n_k} converges uniformly to d. This proves the claim.

Before proving that $\sim_\infty \supset \sim$, let us make some definitions. Let $I, J \subset [0,1]$ be disjoint closed intervals, and let A be the closure of the open interval in between I and J. We say that the pair I, J is *simply glued* by e if $e^\downarrow(A) - e^\downarrow(I) > 0$ and $e^\downarrow(A) - e^\downarrow(J) > 0$. Recall that if A is a compact set then $e^{\downarrow}(A)$ is the minimum of e on A.

The terminology comes from the following two observations.

1. If I, J are simply glued then there are infinitely many pairs $(x_1, x_2) \in I \times J$ such that $x_1 \sim x_2$. To see this, let $h \in (e^{\downarrow}(I), e^{\downarrow}(A)) \cap (e^{\downarrow}(J), e^{\downarrow}(A))$ be an arbitrary point in the intersection. Let $x_1 = \max\{x \in I : e(x) = h\}$ and $x_2 = \min\{x \in J : e(x) = h\}$, these are well defined by the intermediate value theorem. Then $x_1 \sim x_2$. Varying h gives the desired infinite collection of pairs.

2. If an equivalence class $[x]$ of $\sim$ has 2 or 3 elements, then any pair of disjoint intervals containing x_1 and x_2 respectively are simply glued by $\sim$. This can be seen by considering the cases when $|[x]| = 2$ and $|[x]| = 3$ seperately.

The point is that 'simple gluing' is preserved under uniform limits of the excursion, and we can now prove that $\sim_\infty \supset \sim$. Suppose $(x, y) \in \sim$, Let I and J be closed balls of radius ϵ centered at x and y respectively, then since the degree of $\sim$ is bounded by 3, I and J are simply glued by e. It is easy to see then that I and J are simply glued by e_{n_k}, for sufficiently large n_k. It follows that there exists x_{n_k}, y_{n_k} in I and J respectively such that $x_{n_k} \sim_{n_k} y_{n_k}$. By construction, $|x - x_{n_k}| < \epsilon$ and $|y - y_{n_k}| < \epsilon$, and since ϵ was arbitrary, we get $(x, y) \in \sim_\infty$. $\square$

As a consequence of this Lemma, we get that the laminations induced by the welding map converge in law to the Brownian lamination.

Corollary 4.4.2. *Let $\iota_n : \mathbb{D}^* \to \mathbb{C}$ be the conformal welding map for a uniform random arc pairing lamination with n edges. Let $\sim_{\iota_n}$ be the associated lamination. Then $\sim_{\iota_n}$ converges in law to the Brownian lamination.*

Now we wish to show that the subsequential limits of the ι_n solve the Brownian lamination. First we need to know that continuous maps induce laminations.

Lemma 4.4.3 ([57, Proposition II.3.3]). *Suppose $f : \mathbb{D}^* \to \mathbb{C}$ is homeomorphism and extends continuously to $\overline{\mathbb{D}^*}$. Then $\sim_f$ is a lamination. Here $\sim_f$ is the equivalence relation $x \sim_f y \iff f(x) = f(y)$.*

It is an easy exercise to show that if ι_n converges uniformly to ι then any subsequential limit $\sim_\infty$ of the induced laminations $\sim_{\iota_n}$ must contain the lamination of the limit $\sim_\iota$. The reverse inclusion $\sim_\infty \supset \sim_\iota$ is not true in general so we cannot say that $\sim_\infty = \sim_\iota$ and hence $\sim_n \to \sim_\iota$. The next two lemmas allow us to get around this issue. The first lemma shows that, in our setting, the lamination of the limit, $\sim_\iota$, must have finite equivalence classes. The second lemma shows that no chords can be added to the limit of the laminations, $\sim_\infty$, without creating an infinite equivalence class. Together with the aforementioned fact that $\sim_\infty \subset \sim_\iota$, this proves that $\sim_\infty = \sim_\iota$.

Proposition 4.4.4. *Suppose $f : \mathbb{D}^* \to \mathbb{C} \backslash E$ is conformal and extends continuously to the boundary. Suppose f is Hölder continuous on $\mathbb{T}$ with exponent α. Then $|f^{-1}(p)| \leq 2/\alpha$ for all $p \in f(\overline{\mathbb{D}})$. In particular if $\sim$ is the lamination induced by f then $\sim$ has finite equivalence classes.*

Note that the bound is sharp, as can be seen by considering the conformal maps to the exterior of the 'stars' $K_m := \{z : z^m \in [0, 1]\}$.

Proof. Let m be a finite integer such that $m \leq |f^{-1}(p)|$ (this is to circumvent any difficulties that might occur if $|f^{-1}(p)| = \infty$). For sufficiently small $r > 0$, the preimage $f^{-1}(B_r(p)) \cap \mathbb{T}$ consists of at least m disjoint arcs, each containing a point of $f^{-1}(p)$. Suppose f is (C, α)-Hölder continuous on $\mathbb{T}$. Then the diameter of each arc is greater than $(r/C)^{1/\alpha}$. Using the Lebesgue measure in (2.2), it is a straightforward computation to show that the logarithmic capacity of $f^{-1}(B_r(p))$ is bounded below by $c_1 r^{1/(m\alpha)}$ for some constant c_1 that does not depend on r. Let Γ be the family of paths in $\mathbb{D}^*$ joining $\{z : |z| = 2\}$ to $f^{-1}(B_r(p)) \cap \mathbb{T}$. By Pfluger's Theorem 2.3.6 and the preceding observation about the capacity, we have that the extremal length of Γ is (up to an additive constant) bounded above by $\frac{1}{\pi} \log(r^{-1/(m\alpha)})$

On the other hand, we know by conformal invariance that the extremal length of Γ is (up to an additive constant) bounded below by $\frac{1}{2\pi}\log\left(r^{-1}\right)$. Taking $r \to 0$ shows that $m \leq \frac{2}{\alpha}$ as desired. $\square$

Recall from the comment after Lemma 4.4.1 that if ι_n is the welding map for the uniform random tree then $\sim_{\iota_n}$ converges to the Brownian lamination.

Lemma 4.4.5. *Let $\sim$ be the Brownian lamination. Almost surely, it is 'infinitely maximal' in the following sense: every chord in $\mathbb{T}$ not in $\sim$ crosses infinitely many chords in $\sim$.*

Proof. Suppose $x \nsim y$, then $d_{\mathbb{e}}(x,y) > 0$. Without loss of generality assume $x < y$. There are three cases to consider. If $\mathbb{e}(x) > \mathbb{e}(y)$, then for each $\mathbb{e}(y) < h < \mathbb{e}(x)$ we can define $a_1 = \max_{t<\mathbb{e}(x)} \mathbb{e}(t) = h$ and $a_2 = \min_{t\in(\mathbb{e}(x),\mathbb{e}(y))} \mathbb{e}(t) = h$, then $a_1 \sim a_2$ and this chord crosses (x,y). Since h was arbitrary, this gives infinitely chords (a_1,a_2) that cross (x,y).

The case $\mathbb{e}(x) < \mathbb{e}(y)$ is similar.

On the other hand, suppose $\mathbb{e}(x) = \mathbb{e}(y)$, then we must have $m = \mathbb{e}^{\downarrow}([x,y]) < \mathbb{e}(y)$. For $m < h < \mathbb{e}(y)$, we can define $a_1 = \max_{t\in(\mathbb{e}(x),\mathbb{e}(y))} \mathbb{e}(t) = h$ and $a_2 = \min_{t>\mathbb{e}(y)} \mathbb{e}(t) = h$. Then $a_1 \sim a_2$ and this chord crosses (x,y). Since h was arbitrary, this gives infinitely chords (a_1,a_2) that cross (x,y). $\square$

Now we can show that $\sim_{\iota_n}$ converges to the Brownian lamination.

Proof of Theorem 1.3.2. The space $\mathcal{K}(\mathbb{T}^2)$ is compact, and we have shown that μ_n is tight on $\mathbb{C}(\mathbb{T})$. Therefore $\tilde{\mu}_n$ is tight on $\mathcal{K}(\mathbb{T}^2) \times \mathbb{C}(\mathbb{T})$, and there is a sequence $\tilde{\mu}_{n_k}$ that converges in distribution to a measure $\tilde{\mu}_\infty$ on $\mathcal{K}(\mathbb{T}^2) \times \mathbb{C}(\mathbb{T})$.

By the Skorohod representation theorem, there is a coupling of $(\sim_{n_k}, \iota_{n_k})$ and $(\sim_\infty, \iota)$ such that $(\sim_{n_k}, \iota_{n_k}) \to (\sim_\infty, \iota)$ almost surely, Moreover, by Lemma 4.4.1, the distribution of $\sim_\infty$ is that of a Brownian lamination. Furthermore, $\sim_\iota$ must contain $\sim_\infty$ because ι_{n_k} converges uniformly to ι. But $\sim_\infty$ is maximal in the sense that if $\sim_\iota$ has strictly more chords than $\sim_\infty$ then $\sim_\iota$ has an equivalence class with infinitely many points (Lemma 4.4.5). However, $\sim_\iota$ has finite equivalence classes by Proposition 4.4.4, so we have actually $\sim_\iota=\sim_\infty$. By the main result of [36], there is only one measure on pairs $(\sim_\infty, \iota)$ such that $\sim_\iota = \sim_\infty$. It follows that $(\sim_n, \iota_n)$ converges to $(\sim_\infty, \iota)$ in law. $\square$

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Appendix A

RANDOM WALK ESTIMATES

The estimates in this chapter are needed for Section 4. See in particular Section 4.2.1 for explanations of the notation used here.

The following lemma will be used to convert probabilistic statements about the maximum and modulus of continuity of bridge walks to corresponding statements about walks conditioned to be positive. Note that the statement is meaningful only when $w-g$ is even and $g \geq 1$. The $g = 1$ case is special because then the left hand side can be identified with $\{0, \ldots, w-1\} \times \mathbf{E}_{w-1}(0)$.

Lemma A.0.1. *There is a bijection*

$$\varphi : \{0, \ldots, w-1\} \times \mathbf{Z}_w(0 \uparrow g) \to \{1, \ldots, g\} \times \mathbf{W}_w(0 \to g) \tag{A.1}$$

Moreover, the mapping preserves the maximum and modulus of continuity in the following sense: if $\varphi(t, S) = (y, \tilde{S})$, then

1. *If $\mathbb{e}_{\tilde{S}}$ is $(L, 1/3)$-Hölder continuous the $\mathbb{e}_S$ is $(2L, 1/3)$-Hölder continuous.*

2. $\frac{1}{3} \max |S| \leq \max |\tilde{S}| \leq 3 \max |S|$.

Recall that $\mathbb{e}_S : [0,1] \to \mathbb{R}$ is the Brownian rescaling of S, $\mathbb{e}_S(t) = w^{-1/2} S(wt)$.

The existence of the bijection is known as the Dvoretzky-Motzkin cycle lemma [23], but the other two properties are usually not stated in the literature so we present the proof here.

Proof. First observe that there is a natural action of the cyclic group $\mathbb{Z}_w = \{0, \ldots, w-1\}$ on $\mathbf{W}_w(0 \to g)$, defined by cyclically permuting the increments of the walks. More formally, if $t_0 \in \mathbb{Z}_w$ and $S \in \mathbf{W}_w(0 \to g)$, define

$$\mathcal{C}_{t_0} S(t) = \begin{cases} S(t_0 + t) - S(t_0) & \text{if } 0 \le t \le w - t_0 \\ S(t + t_0 - w) + g - S(t_0) & \text{if } w - t_0 \le t \le w. \end{cases} \tag{A.2}$$

Now we describe the map φ. Suppose $(t_0, S) \in \{0, \ldots, w-1\} \times \mathbf{Z}(0 \uparrow g)$. Let $y_0 = g - \min S|_{[t_0,w]}$. Since for $S \in \mathbf{Z}(0 \uparrow g)$, the last step is always going up from $g-1$ to g, we have $\min S|_{[t_0,w]}| \le g - 1$ and hence $y_0 \in \{1, \ldots, g\}$. Then $\varphi(t_0, S) = (y_0, \mathcal{C}_{t_0} S)$ is the desired mapping.

The inverse mapping is described as follows. Suppose $(y_0, \tilde{S}) \in \{1, \ldots, g\} \times \mathbf{W}_w(0 \to g)$. Let $h_0 = \min \tilde{S} + y_0$. Let $s_0 = \max\{s : \tilde{S}(s) = h_0 - 1\} + 1$. Then $\tilde{S}(s_0) = h_0$ and $\varphi^{-1}(y_0, \tilde{S}) = (w - s_0, \mathcal{C}_{s_0} \tilde{S})$ is the desired inverse mapping, and the bijection is proved.

Statement 1) follows from the fact that if $\mathbb{e}_S$ is (L, α)-Hölder continuous then for any t_0, $\mathbb{e}_{\mathcal{C}_{t_0} S}$ is $(2L, \alpha)$-Hölder continuous. Statement 2) follows from the fact that the maximum of any cyclic permutation of any walk $S \in \mathbf{W}_w(0 \to g)$ is bounded by $(g - \min S) + \max S$ which is in turn bounded by $3 \max |S|$. $\square$

We will need the following asymptotics for the number of $\mathbf{Z}$ walks in Lemma 4.3.8.

Corollary A.0.2. *We have, for integers $a, g > 0$*

$$|\mathbf{Z}_a(0 \uparrow g)| = \frac{g}{a} \binom{a}{\frac{a}{2} + \frac{g}{2}},$$

and

$$|\mathbf{Z}_a(0 \uparrow g)| \cdot 2^{-a} \le C_{\text{stir}} g^{-2} (a/g^2)^{-3/2} e^{-\frac{g^2}{3a}} \tag{A.3}$$

for some constant C_{stir}.

Proof. The equality follows immediately from (A.1). For the inequality, we use the simple consequence of Stirling's formula

$$C_{\text{stir}}^{-1} w^{-1/2} \le \binom{w}{w/2} 2^{-w} \le C_{\text{stir}} w^{-1/2} \tag{A.4}$$

and obtain

$$|\mathbf{Z}_a(0 \uparrow g)| \cdot 2^{-a} = \frac{g}{a}\binom{a}{\frac{a}{2}+\frac{g}{2}}2^{-a} = \frac{g}{a}\binom{a}{\frac{a}{2}}2^{-a} \cdot \frac{(a/2)(a/2-1)\cdots(a/2-g/2+1)}{(a/2+1)(a/2+2)\cdots(a/2+g/2)}$$
$$\leq C_{\mathrm{stir}} g a^{-3/2}\left(\frac{1}{1+\frac{g}{a}}\right)^{g/2} \leq C_{\mathrm{stir}} g^{-2}(a/g^2)^{-3/2} \cdot \exp\left(-\frac{g^2}{3a}\right),$$

where in the last equality we have used the fact that $y\log(1+x) \geq \frac{2}{3}xy$ when $x \in [0,1]$ and $y \geq 0$. $\square$

The next lemma allows us to convert probabilistic statements about walks to probabilistic statements about bridges, and vice versa.

Lemma A.0.3 (Local absolute continuity of bridges and walks)**.** *Fix $0 < u < w$ integer, suppose $|h| \leq c_0 w^{1/2}$, and let A be a subset of $\mathbf{W}_u(0)$. Suppose $\frac{u}{w} \leq \frac{3}{4}$. Then $\mathbb{P}^{\mathbf{W}_w(0\to h)}(S|_{[0,u]} \in A) \leq C_0 \mathbb{P}^{\mathbf{W}_w(0)}(S|_{[0,u]} \in A)$, where the constant C_0 only depends on c_0.*

If, in addition, there exists $c_1 \leq 1$ such that A only contains walks for which $S(u) \leq c_1(w-u)^2$, then $\mathbb{P}^{\mathbf{W}_w(0)}(S|_{[0,u]} \in A) \leq C_0 \mathbb{P}^{\mathbf{W}_w(0\to 0)}(S|_{[0,u]} \in A)$ where C_0 only depends on c_1.

Proof. The first statement is proved for $h = 0$ in [38, Lemma 3], and the statement for general h follows from the obvious modifications. It suffices to consider the case when A has only one element, $A = \{S'\}$ and then the relevant probabilities can be written down explicitly in terms of $S'(u)$. The second statement follows from the same proof. $\square$

For example, the previous lemma allows us to deduce the Hölder continuity of the Brownian rescaling (4.6) of $\mathbf{Z}$-walks:

Lemma A.0.4. *For $\epsilon > 0$, $\Lambda, L_2 > 1$, there exists $L_3 > 0$ large such that the following holds. If $g \geq \Lambda$ with $g - w$ even and if $g' = \lfloor g/\Lambda \rfloor$ satisfies $wg'^{-2} \in [L_2^{-1}, L_2]$, then*

$$\mathbb{P}(\mathbb{e}_S \textit{ is } (L_3, 1/3) - \textit{Hölder continuous }) \geq 1 - \epsilon$$

where S is a uniform random walk of type $\mathbf{Z}_w(0 \uparrow g', \max < g)$.

Proof. The idea of the proof is to relate walks of this type to walks of type $\mathbf{B}$ using Lemma A.0.1. This relationship essentially preserves the modulus of continuity of the walk. Then we use the fact that walks of type $\mathbf{B}$ are locally absolutely continuous to the simple random walk. The Hölder continuity of the $\mathbf{Z}$ walks then follows from the Hölder continuity of the simple random walk. This sort of argument was used in [38] to get uniform bounds for the maximum of a random walk excursion.

It suffices to prove the result when S is a uniform random walk of type $\mathbf{Z}_w(0 \uparrow g')$, because the uniform measure on $\mathbf{Z}_w(0 \uparrow g', \max \leq g')$ is absolutely continuous to the uniform measure on $\mathbf{Z}_w(0 \uparrow g')$, indeed by Proposition A.1.1 we have $\mathbb{P}_{\mathbf{Z}_w(0 \uparrow g')}(\max S < g) \geq c_0$ for some constant c_0 that only depends on Λ and L_2.

Lemma A.0.1 and its proof implies that we can sample a uniform random element of $\mathbf{Z}_w(0 \uparrow g')$ by choosing a uniform random element of $\mathbf{W}_w(0 \to g')$, and then applying a certain (random) cyclic permutation of the increments, and as observed in that lemma, this cyclic permutation preserves the modulus of continuity. Therefore it suffices to prove the result for uniform random walks of type $\mathbf{W}_w(0 \to g')$. Now observe from the triangle inequality that if a function is $(L/2, \alpha)$-Hölder continuous when restricted to $[0, 1/2]$ and $[1/2, 1]$ respectively, then it is (L, α)-Hölder continuous on $[0, 1]$. Therefore from symmetry and the union bound it suffices to find L_3 large enough that

$$\mathbb{P}_{\mathbf{W}_w(0 \to g)}(\mathbb{e}_S|_{[0,1/3]} \text{ is not } (L_3/2, 1/3) - \text{Hölder continuous}) \leq \epsilon/2.$$

By Lemma A.0.3 below, the law of $S|_{[0, \lceil w/2 \rceil]}$ under $\mathbb{P}_{\mathbf{W}_w(0 \to g')}$is absolutely continuous to the law of $S|_{[0, \lceil w/2 \rceil]}$ under $\mathbb{P}_{\mathbf{W}_w(0)}$ with a constant C_0 that only depends on L_2. So it suffices to find L_3 large enough that

$$\mathbb{P}_{\mathbf{W}_w(0)}(\mathbb{e}_S|_{[0,1/2]} \text{ is not } (L_3/2, 1/3) - \text{Hölder continuous}) \leq \epsilon/(2C_0).$$

This last statement follows from the proof of Kolmogorov's continuity criterion. $\square$

A.1 Bounds on the extrema of a random walk

In this section we collect some bounds on the probability that a random walk of length w exceeds, or does not exceed, a given height y. The analogous bounds for Brownian motion are simpler to state and prove, but we need statements that are uniform in w and y.

Proposition A.1.1. *a) We have, for some $c_0 > 0$,*

$$\mathbb{P}_{\mathbf{W}_w(0)}(\max|S| \geq y) \lesssim \exp\left(-\frac{2y^2}{w}\right),$$

$$\mathbb{P}_{\mathbf{E}_w(0\to 0)}(\max|S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right),$$

and

$$\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\max|S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right),$$

$$\mathbb{P}_{\mathbf{W}_w^{\updownarrow}(0\to 0)}(\max|S| \geq y) \lesssim \exp\left(-c_0\frac{y^2}{w}\right).$$

b) For $\delta > 0$ there exists $c_\delta > 0$ such that if $w \geq \delta y^2$, then the conditional probabilities of a) are either zero or bounded below by c_δ,

$$\mathbb{P}_{\cdot}(\max S \geq y) \geq c_\delta.$$

c) Finally, there exists $c_0 > 0$ such that the conditional probabilities of a) satisfy

$$\mathbb{P}_{\cdot}(\max|S| \leq y) \leq \frac{3}{2}\exp\left(-c_0\frac{w}{y^2}\right).$$

Proof. By André's reflection principle [25, page 72],

$$\mathbb{P}_{\mathbf{W}_w(0)}(\max S \geq y) = 2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) > y) + \mathbb{P}_{\mathbf{W}_w(0)}(S(w) = y)$$

so that

$$2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) > y) \leq \mathbb{P}_{\mathbf{W}_w(0)}(\max S \geq y) \leq 2\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq y). \tag{A.5}$$

Now $\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq y) \leq \exp(-2y^2/w)$ by Hoeffding's inequality [33, Theorem 2] and by the union bound we have proved the first claim $\mathbb{P}_{\mathbf{W}_w(0)}(\max|S| \geq y) \leq 4\exp(-2y^2/w)$.

On the other hand, by the central limit theorem we have that

$$\mathbb{P}_{\mathbf{W}_w(0)}(S(w) \geq \delta^{-1/2} w^{1/2}) \to c'_\delta > 0 \text{ as } w \to \infty \tag{A.6}$$

and claim b) for $\mathbf{W}_w(0)$ follows. Claim c) for $\mathbf{W}_w(0)$ now follows from the strong Markov property by decomposing the walk into subwalks of length proportional to y^2 and then using the result of part b) on each of these walks: if any subwalk varies more than $2y$ from its initial point, then the maximum absolute value of the walk must exceed y. This shows that the probability that the maximum is bounded by y is less than $C_0 \exp\left(-c_0 \frac{w}{y^2}\right)$. The constant C_0 may be taken to be $\frac{3}{2}$ by taking c_0 smaller if necessary.

A coupling argument similar to the proof of Lemma A.2.1 below shows that the absolute maximum of a bridge is stochastically dominated by the absolute maximum of a Bernoulli walk, so this proves a) for bridges $\mathbf{W}_w(0 \to 0)$. The exact same proof works for $\mathbf{W}^{\uparrow}_w(0 \to 0)$ and $\mathbf{W}_w(0 \to 1)$.

This latter statement can be used together with the cycle lemma, Lemma A.0.1, to prove part a) for $\mathbb{P}_{\mathbf{E}_w(0)}$.

Part b) for bridges and excursions is proved similarly to part b) for walks, and follows from the fact that the measures converge to the Brownian bridge and Brownian excursion respectively.

Part c) for bridges follows from part c) for walks together with Lemma A.0.3 above, which says that the initial part of a random bridge is almost indistinguishable from the initial part of a random walk. Part c) for excursions then follows from the cycle lemma. □

The following bounds are useful when wg^{-2} is large. In particular, the second bound does not degenerate even when $wg^{-2} \to \infty$.

Lemma A.1.2. *For integers $w, g > 0$,*

$$\mathbb{P}_{\mathbf{W}^{\uparrow}_w(0\to 0)}(\min S \geq -g) \leq 4w^{-1}g^2. \tag{A.7}$$

For $\epsilon > 0$, we have for sufficiently large μ that, for all w, g,

$$\mathbb{P}_{\mathbf{W}_w(\mu g \to \mu g)}(\min S > g | \min S \geq 0) \geq 1 - \epsilon.$$

Proof. We begin with the first inequality. The statement is vacuously true for $w^{-1}g^2 > 1/4$, so in what follows we can assume in particular that $\frac{g}{w/2} \leq 1/2$.

Recall that $\mathbf{W}_w^{\uparrow}(0 \to 0)$ maps bijectively onto $\mathbf{W}_{w-2}(0 \to 0)$, and this map can be realized by forgetting the first and last steps of the walk and translating the whole walk up one unit. Therefore

$$\mathbb{P}_{\mathbf{W}_w^{\uparrow}(g\to g)}(\min S \geq 0) = \mathbb{P}_{\mathbf{W}_w^{\uparrow}(0\to 0)}(\min S \geq -g) = \mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S \geq -g+1). \tag{A.8}$$

We have $|\mathbf{W}_{w-2}(0 \to 0)| = \binom{w-2}{w/2-1}$, and by André's reflection principle, $|\mathbf{W}_{w-2}(0 \to 0, \min S < -g+1)| = |\mathbf{W}_{w-2}(0 \to -2g)| = \binom{w-2}{w/2+g-1}$. Therefore

$$\begin{aligned}
\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S < -g+1) &= \left(\frac{w/2-g}{w/2}\right)\left(\frac{w/2-g+1}{w/2+1}\right)\cdots\left(\frac{w/2-1}{w/2+g-1}\right)\\
&\geq \left(1-\frac{g}{w/2}\right)^g\\
&\geq e^{-4g^2/w} \qquad\qquad (A.9)\\
&\geq 1-4g^2/w
\end{aligned}$$

where in the last two inequalities we have used the facts that $1-x \geq e^{-2x}$ for $x \in [0, 1/2]$, and $e^{-x} \geq 1-x$.

Together with (A.8), this proves the first inequality of the lemma.

Notice that the derivation leading up to (A.9) actually shows that for $\theta > 0$ there exists $M > 0$ such that

$$\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S \geq -g+1) \geq e^{-\frac{(2+\theta)g^2}{w}} \qquad \text{whenever } \frac{g}{w/2} \leq \frac{2}{M+1}, \tag{A.10}$$

because for $\theta > 0$ there exists $M > 0$ such that $1-x \geq e^{-(1+\theta)x}$ for $x \in [0, 2/(M+1)]$.

We also have the upper bound

$$\begin{aligned}
\mathbb{P}_{\mathbf{W}_{w-2}(0\to 0)}(\min S < -g+1) &= \left(\frac{w/2-g}{w/2}\right)\left(\frac{w/2-g+1}{w/2+1}\right)\cdots\left(\frac{w/2-1}{w/2+g-1}\right)\\
&\leq \left(1-\frac{g}{w/2+g-1}\right)^g\\
&\leq e^{-\frac{g^2}{w/2+g-1}}. \qquad\qquad (A.11)
\end{aligned}$$

For the second inequality, fix $\theta > 0$ small and $M > 1$ large such that $\frac{1}{2+\theta} \cdot \frac{1}{1/2+(M+1)^{-1}} \geq 1-\epsilon/2$ and such that (A.9) holds. First consider the case $w^{-1}g^2 > \frac{1}{2M\mu}$. Then by Proposition A.1.1, we have for sufficiently large μ,

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g) &= \mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g)\\ &\geq 1 - 4e^{-\frac{(\mu-1)^2 g^2}{2w}} \geq 1 - 4e^{-\frac{(\mu-1)^2}{4M\mu}} \geq 1-\epsilon.\end{aligned}$$

Now suppose $w^{-1}g^2 \leq \frac{1}{2M\mu}$ so that in particular $\frac{\mu g}{w/2} \leq 1/M$ and so $\frac{\mu g+1}{w/2+1} \leq 2/(M+1)$. We have

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g|\min S \geq 0) &= \frac{\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g)}{\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S \geq 0)}\\ &= \frac{\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g)}{\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S \geq -\mu g)}.\end{aligned}$$

Now, (A.10) implies that

$$\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S \geq -\mu g) = \mathbb{P}_{\mathbf{W}_{(w+2)-2}(0\to 0)}(\min S \geq -(\mu g+1)+1) \leq 1 - e^{-(2+\theta)\frac{(\mu g+1)^2}{w+2}},$$

On the other hand, (A.11) gives

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(0\to 0)}(\min S > -(\mu-1)g) &= 1 - \mathbb{P}_{\mathbf{W}_{(w+2)-2}(0\to 0)}(\min S \leq -((\mu-1)g-1)+1)\\ &\geq 1 - e^{-\frac{((\mu-1)g-1)^2}{w/2+(\mu-1)g-1}}\end{aligned}$$

and we get

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(\mu g\to\mu g)}(\min S > g|\min S \geq 0) &\geq \frac{1 - e^{-\frac{((\mu-1)g-1)^2}{w/2+(\mu-1)g-1}}}{1 - e^{-(2+\theta)\frac{(\mu g+1)^2}{w+2}}}\\ &\geq \frac{((\mu-1)g-1)^2}{(\mu g+1)^2} \cdot \frac{1}{2+\theta} \cdot \frac{w+2}{w/2+(\mu-1)g-1}\\ &\geq \frac{((\mu-1)g-1)^2}{(\mu g+1)^2} \cdot \frac{1}{2+\theta} \cdot \frac{w+2}{w/2+(w+2)(M+1)^{-1}}\\ &\geq \frac{\mu^2}{(\mu+1)^2} \cdot \frac{1}{2+\theta} \cdot \frac{1}{1/2+(M+1)^{-1}}\\ &\geq \frac{\mu^2}{(\mu+1)^2}(1-\epsilon/2).\end{aligned}$$

where in the second inequality we have used the fact that $\frac{1-e^{-x}}{1-e^{-y}} \geq \frac{x}{y}$ when $x \leq y$, which in turn follows from the fact that $x \mapsto \frac{1-e^{-x}}{x}$ is decreasing. In the third inequality we have used the fact that $\frac{(\mu-1)g-1}{w/2+1} \leq 2(M+1)^{-1}$. The fourth inequality is true for $g \geq 1$ and can be seen be differentiating both sides with respect to g.

For sufficiently large μ, this last expression is greater than $1-\epsilon$, as desired. $\square$

Finally, the next lemma says that the bounds of Proposition A.1.1 still hold even when we condition on certain events.

Lemma A.1.3. *Let S be a uniformly random walk of type $\mathbf{W}_w(g \to g, \min \geq 0)$. Then* $\mathbb{P}(\max S \geq \mu g) \lesssim e^{-c_0(\mu-1)^2 g^2/w}$.

Similarly, $\mathbb{P}_{\mathbf{W}_w(\mu g \to \mu g)}(\min S \leq g | \min S \geq 0) \lesssim e^{-c_0(\mu-1)^2 g^2/w}$.

Proof. By translation invariance, Lemma A.2.4 below, and Proposition A.1.1,

$$\begin{aligned}\mathbb{P}_{\mathbf{W}_w(g\to g)}(\max S \geq \mu g | \min S \geq 0) &= \mathbb{P}_{\mathbf{W}_w(0\to 0)}(\max S \geq (\mu-1)g | \min S \geq -g) \\ &\leq \mathbb{P}_{\mathbf{E}_w(0\to 0)}(\max S \geq (\mu-1)g) \lesssim \exp\left(-c_0\frac{(\mu-1)^2 g^2}{w}\right)\end{aligned}$$

and this proves the first statement.

The second statement follows from a similar argument, using Lemma A.2.5 below and Proposition A.1.1. $\square$

A.2 Monotonicity properties of conditioned random walks

The next few lemmas make precise some intuitively clear monotonicity relations between various types of walks.

Lemma A.2.1 (Strong Monotonicity)**.** *Suppose $w > 0$ is even and fix a partition of $[0, w]$ into almost disjoint closed intervals $A_1, \ldots, A_m$ with endpoints that are even integers. Let S be a uniformly random walk of type $\mathbf{E}_w(0 \to 0)$. For $i = 1, \ldots, m$ let $\tilde{S}_i$ be a uniformly random element of $\mathbf{E}_{A_i}(0 \to 0)$. Let $\tilde{S}$ be the concatenation of the S_i, so that $\tilde{S} \in \mathbf{E}_w(0)$. Then* $\max S \succ \max \tilde{S} = \max \tilde{S}_1 \vee \cdots \vee \max \tilde{S}_m$.

Proof. For $i = 1, \ldots, m$, let $I_i = [\tau_i^-, \tau_i^+]$ where τ_i^-, τ_i^+ is the first and last time respectively that S intersects S_i. In the case that these times do not exist, let $I_i = \emptyset$.

Conditioned on $\{I_i\}_{i=0}^m$ and the values of S at the endpoints of the I_i, the collection of walks $\{S|_{I_i}\}_i$ has the distribution of $m+1$ independent bridges (conditioned to be nonnegative). Likewise, $\{\tilde{S}|_{I_i}\}_i$ has the same distribution.

Thus if we define S' to be the excursion obtained by replacing the part of S on each I_i with the corresponding part of $\tilde{S}$, then S' has uniform distribution on $\mathbf{E}_w(0)$. On the other hand, $S|_{A_i \setminus I_i} \geq \tilde{S}|_{A_i \setminus I_i}$ pointwise, due to the boundary conditions of $\tilde{S}|_{A_i}$. It follows that $S' \geq \tilde{S}$ pointwise, and so $(S', \tilde{S})$ is a coupling which proves the desired result. □

Lemma A.2.2. *Let S be a uniform random element of $\mathbf{E}_w(0)$ and let $\tilde{S}$ be a uniform random element of $\mathbf{Z}_{w+1}(0 \uparrow g)$, where $g \geq 1$. Then $\max \tilde{S} \succ \max S$.*

Proof. The proof is the similar to the proof of Lemma A.2.1. □

Lemma A.2.3 (Monotonicity)**.** *For all integer $g > 0$ and $w_2 \geq w_1$ even,*

$$\mathbb{P}_{\mathbf{E}_{w_2}(0\to 0)}(\max \geq g) \geq \mathbb{P}_{\mathbf{E}_{w_1}(0\to 0)}(\max \geq g).$$

Proof. Take $I_1 = [0, w_1]$ and $I_2 = [w_1, w_2]$ in Lemma A.2.1. □

Lemma A.2.4. *Fix $g \geq 0$. Let S be a uniformly random element of $\mathbf{E}_w(0)$ and let $\tilde{S}$ be a uniformly random element of $\mathbf{W}_w(0 \to 0, \min \geq -g)$. Then $\max S \succ \max \tilde{S}$.*

Proof. Let $A \subset [0, w]$ be the set of times for which (the linear interpolation of) $\tilde{S}$ is strictly negative. Conditioned on A, the distribution of $\tilde{S}|_{[0,w]\setminus \overline{A}}$ is that of independent excursions over each component of $[0, w]\setminus \overline{A}$. It follows from Lemma A.2.1 that $\max S \succ \max \tilde{S}$. □

Lemma A.2.5. *Fix $w > 0$ even, $g \leq 0$ and $h \geq g$. Let S be a uniform random walk in $\mathbf{W}_w(h \to 0)$ and let $\tilde{S}$ be the same object but conditioned on $\min S \geq g$. Then $\tilde{S} \succ S$.*

Proof. We have

$$\mathbb{P}(\min S \geq g | S(1) = h+1) \geq \mathbb{P}(\min S \geq g).$$

First notice that

$$\mathbb{P}_{\mathbf{W}_{w(h\to 0)}}(\min S \geq g | S(1) = h+1) \geq \mathbb{P}_{\mathbf{W}_{w(h\to 0)}}(\min S \geq g). \tag{A.12}$$

This statement can be proved by a coupling argument. We can couple the two types of walks by running them independently until they meet, then making them equal to each each other. In this coupling, the walk satisfying $S(1) = h + 1$ will always dominate the other walk, and this proves the statement.

On the other hand, using the statement with Bayes' rule gives

$$\mathbb{P}_{\mathbf{W}_{w(h\to 0)}}(S(1) = h+1 | \min S \geq g) \geq \mathbb{P}_{\mathbf{W}_{w(h\to 0)}}(S(1) = h+1). \tag{A.13}$$

Using this, we get a coupling of $\tilde{S}$ and S for which $\tilde{S} \geq S$: We ensure that $\tilde{S}(1) \geq S(1)$ using (A.13), then we let the walks evolve independently until they meet again, and then we use (A.13) again, and so on. $\square$

Appendix B

LARGE DEVIATIONS FOR MARKOV CHAINS

The results of this section can be found in [20]. For the reader's convenience, we give a self contained presentation.

Let $\omega_1, \omega_2, \ldots$ be a Markov chain on a state space Ω with transition kernels $\pi(x, dy)$. Let $u : \Omega \to \mathbb{R}$ be a function. For each $x, y \in \Omega \times \Omega$, let $f(x, y) = \frac{u(y)}{\int u(z)\pi(x,dz)}$.

Lemma B.0.1. *For each $n \geq 1$ and any choice of ω_1, we have*

$$\mathbb{E}\left(f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)\right) = 1.$$

Proof. We have, by the tower property and the Markov property,

$$\begin{aligned}
\mathbb{E}\left(f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)\right) &= \mathbb{E}[\mathbb{E}[f(\omega_1, \omega_2)f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)|(\omega_1, \omega_2)]] \\
&= \mathbb{E}[f(\omega_1, \omega_2)\mathbb{E}[f(\omega_2, \omega_3)\cdots f(\omega_{n-1}, \omega_n)|\omega_2]] \\
&= \mathbb{E}[f(\omega_1, \omega_2)] \\
&= 1
\end{aligned}$$

where the second last equality is by induction on n. $\square$

Now let Γ be a set of probability measures on Ω. For $u : \Omega \to (0, \infty)$, let $\lambda_u : \Omega \to \widetilde{R}$ be the function

$$\lambda_u(x) = \log\left(\frac{u(x)}{\int u(y)\pi(x, dy)}\right).$$

For $x \in \Omega$, let δ_x denote the Dirac mass at x.

Theorem B.0.2. *For any $u : \Omega \to (0, \infty)$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} \delta_{\omega_k} \in \Gamma\right) \leq \frac{\mathbb{E}u(\omega_1)}{\inf u} \exp\left(-n \inf_{\mu \in \Gamma} \int \lambda_u d\mu\right). \tag{B.1}$$

Proof. Observe that

$$\mathbb{1}\left\{\frac{1}{n}\sum_{k=1}^{n}\delta_{\omega_k}\in\Gamma\right\}\le\exp\left(\frac{1}{n}\sum_{k=1}^{n}\lambda_u(\omega_k)-\inf_{\mu\in\Gamma}\int\lambda_u d\mu\right)$$

because the term in the parentheses is positive whenever the expression on the left is equal to 1. Taking expectations of the $n-$th power of both sides yields

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{\omega_k}\in\Gamma\right)\le\exp\left(-n\inf_{\mu\in\Gamma}\int\lambda_u d\mu\right)\mathbb{E}\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right). \tag{B.2}$$

We have

$$\begin{aligned}\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right)&=\prod_{k=1}^{n}\frac{u(\omega_k)}{\int u(y)\pi(\omega_k,dy)}\\&=\frac{u(\omega_1)}{\int u(y)\pi(\omega_n,dy)}\prod_{k=1}^{n-1}\frac{u(\omega_{k+1})}{\int u(y)\pi(\omega_k,dy)}.\end{aligned}$$

Thus

$$\mathbb{E}\exp\left(\sum_{k=1}^{n}\lambda_u(\omega_k)\right)\le\mathbb{E}\frac{u(\omega_1)}{\inf u}\mathbb{E}\prod_{k=1}^{n-1}\frac{u(\omega_{k+1})}{\int u(y)\pi(\omega_k,dy)}.$$

The expectation of the product is equal to 1, by Lemma B.0.1. Substituting this into (B.2) yields the result. $\square$

Theorem B.0.3. *Let $A\subset\Omega$ be a subset. Suppose $u:\Omega\to[1,\infty)$ is a function with $\lambda_u\ge 0$. Then for each $\epsilon>0$,*

$$\mathbb{P}\left(\frac{1}{n}\left|\{k:\omega_k\in A\}\right|\ge\epsilon\right)\le\mathbb{E}u(\omega_1)\exp\left(-n\epsilon\inf_{\omega\in A}\lambda_u(\omega)\right). \tag{B.3}$$

Proof. We apply Theorem B.0.2 with $\Gamma=\{\mu:\mu(A)\ge\epsilon\}$. We have $\inf_{\mu\in\Gamma}\int\lambda_u d\mu\ge\inf_{\mu\in\Gamma}\int_A\lambda_u d\mu\ge\mu(A)\inf_{\omega\in A}\lambda_u(\omega)\ge\epsilon\inf_{\omega\in A}\lambda_u(\omega)$ because $\lambda_u\ge 0$. $\square$