1. (2 pts each, 40 pts total) Place the letter corresponding to the correct answer in the box next to each question. Each correct answer is worth 2 points.

(i) \[ \text{Simplify } \ln(x^2e^{2x}) \] (a) \( 2 \ln(x) + 2x \) (b) \( 4x \ln(x) \) (c) \( 2x \ln(x) \) (d) \( 2 \ln(x) + x^2 \) (e) \( 4x^2 \) (f) none of these.

(ii) If \( \frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} \) then \( A = \) (a) 1 (b) 2 (c) 3 (d) 4 (e) 5 (f) none of these.

(iii) With the substitution \( x = 3 \sin \theta \), the integral \( \int \frac{x^2 dx}{\sqrt{9-x^2}} \) becomes (a) \( 9 \int \frac{d\theta}{\sin \theta} \) (b) \( 9 \int \sin^2 \theta d\theta \) (c) \( 9 \int \sin^3 \theta d\theta \) (d) \( 27 \int (1 - \sin \theta) d\theta \) (e) \( 27 \int \sin^3 \theta d\theta \) (f) none of these.

(iv) Which of the following improper integrals converges? (a) \( \int_0^1 \frac{\ln x}{x} dx \) (b) \( \int_0^1 x^{-3} dx \) (c) \( \int_0^1 x^{-1/4} dx \) (d) \( \int_1^\infty \frac{1}{\sqrt{x}} dx \) (e) \( \int_0^1 \frac{1}{x} dx \). (f) none of these.

(v) If the Taylor series \( \sum_{n=1}^\infty c_n(x-2)^n \) converges at \( x = 5 \), then it must also converge at (a) -5 (b) -2 (c) -1 (d) 0 (e) 7 (f) none of these.

(vi) Evaluate \( \int_0^2 \frac{2x dx}{x^2-5} \). (a) \( \ln 2 \) (b) \( -\ln 2 \) (c) 0 (d) -\ln 5 (e) \( \ln 4 \) (f) none of these.

(vii) What is the inverse of \( y = e^{2x+1} \)? (a) \( y = \sqrt{\ln(x)} - 1 \) (b) \( y = \frac{1}{2} \ln(x/e) \) (c) \( y = \ln(\sqrt{x}) \) (d) \( y = \frac{1}{2} \ln(x) + 1 \) (e) \( y = \ln(2x+1) \) (f) none of these.

(viii) The formula for Euler’s method of solving \( y' = f(x, y) \) is (a) \( y_{n+1} = y_n - f(x_n, y_n) \Delta x \) (b) \( y_{n+1} = y_n + f(x_n, y_n) \) (c) \( y_{n+1} = y_n + f(x_n, y_n) \Delta x \) (d) \( y_{n+1} = y_n - f(x_n, y_n) \) (e) \( y_{n+1} = y_n + f(x_n, y_n)(\Delta x)^2 \) (f) none of these.

(ix) Find the solution of the differential equation \( \frac{dy}{dx} = (1 + y^2)e^x \). (a) \( y = \tan x \) (b) \( y = e^{\tan x} \) (c) \( y = 1 + \tan^2 x \) (d) \( y = e^x \) (e) \( y = \tan(e^x) \) (f) none of these.
(x) In an oil refinery, a tank contains 2000 gallons of gasoline that initially has 100 pounds of additive dissolved in it. Gasoline containing 2 pounds of additive per gallon is pumped into the tank at 40 gal/min and the well mixed solution is pumped out at 45 gal/min. If \( y(t) \) is the amount of additive at time \( t \) then \( y \) satisfies which of the following differential equations: (a) \( \frac{dy}{dt} = 40 - \frac{45y}{2000-45t} \) (b) \( \frac{dy}{dt} = 80 - \frac{45y}{2000-45t} \) (c) \( \frac{dy}{dt} = 40 - \frac{45y}{2000-45t} \) (d) \( \frac{dy}{dt} = 80 - \frac{40y}{2000-45t} \) (e) \( \frac{dy}{dt} = 40 - \frac{40y}{2000-45t} \) (f) none of these.

(xi) Which of the following is a true identity for hyperbolic trig functions? (a) \( \sinh 2x = 2 \sinh x \cosh x \) (b) \( \sinh^2 x + \cosh^2 x = 1 \) (c) \( \cosh 2x = \cosh x + \sinh x \) (d) \( \cosh^2 x = \sinh^2 x \) (e) \( \sinh^2 x = \frac{1}{2} \cosh 2x \) (f) none of these.

(xii) The improper integral \( \int_0^\infty (1 + x^2)^{-2} dx \) converges if and only if (a) \( s > 1 \) (b) \( s > 1/2 \) (c) \( s > 0 \) (d) \( s < -1 \) (e) \( s < 0 \) (f) none of these.

(xiii) Which of the following is a geometric series? (a) \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) (b) \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) (c) \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots \) (d) \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \) (e) \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \) (f) none of these.

(xiv) The solution of the differential equation \( y' = y^2 - y - 2 \) with initial condition \( y(0) = 0 \) (a) is constant (b) decreases to \(-1\) (c) decreases to \(-\infty\) (d) increases to \(+\infty\) (f) none of these.

(xv) The sequence \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \) is (a) non-decreasing and convergent (b) non-increasing and convergent (c) non-decreasing and divergent (d) non-increasing and divergent (e) bounded and divergent (f) none of these.

(xvi) Which of the following sequences is not bounded for \( n = 1, 2, 3, \ldots \)? (a) \( \{n^2 2^{-n}\} \) (b) \( \{\sqrt{n+1} - \sqrt{n}\} \) (c) \( \{n^2/(1+n^3)\} \) (d) \( \{n/\ln n\} \) (e) \( \{n^2/(1+n)^2\} \) (f) none of these.

(xvii) Suppose \( a_n = 2a_{n-1} + 1 \). If \( a_1 = 1 \) then \( a_5 = \) (a) 15 (b) 5 (c) 31 (d) 8 (e) 16 (f) none of these.

(xviii) The series \( \sum_{n=1}^\infty (2x+1)^n \) converges exactly for (a) all \( x \) (b) \( -1 < x < 0 \) (c) \( -1 < x < 1 \) (d) \( 0 < x < 1 \) (e) \( 0 < x < 1/2 \) (f) none of these.

(xix) The root test says that if \( a_n \geq 0 \) then \( \sum_{n=0}^\infty a_n \) converges if (a) \( \lim_{n \to \infty} a_{n+1}/a_n < 1 \) (b) \( \lim_{n \to \infty} a_n < 1 \) (c) \( \lim_{n \to \infty} (a_n)^{1/n} < 1 \) (d) \( \lim_{n \to \infty} a_n/a_{n+1} < 1 \) (e) \( \lim_{n \to \infty} n a_n < 1 \) (f) none of these.

(xx) Evaluate the improper integral \( \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \). (a) \( \pi \) (b) \( \pi/2 \) (c) 1 (d) \( 2\pi \) (e) 2 (f) none of these.
2. (1 pt each, 10 pts total) Match each function with its Taylor series expansion.

(i) \( (1 - x)^{-1} \)  
(ii) \( \sin x \)  
(iii) \( \ln(1 + x) \)  
(iv) \( (1 + x)^{1/3} \)  
(v) \( e^x \)  
(vi) \( x^2 \cos x \)  
(vii) \( e^{x^2} \)  
(viii) \( \frac{\ln(1 + x)}{1 + x} \)  
(ix) \( \cos x - \sin x \)  
(x) \( \sqrt{1 + x^2} \)

A 1 + 2x + 4x^2 + 8x^3 + 16x^4 + ...  
B 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + ...  
C x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...  
D 1 + x^2 + x^4 + x^6 + ...  
E 1 + \frac{1}{3}x - \frac{1}{5}x^2 + \frac{10}{17}x^3 - \frac{80}{81}x^4 + ...  
F 1 - x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 - ...  
G 1 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{3}{5}x^5 - \frac{15}{16}x^6 + ...  
H x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + ...  
I 1 + x + \frac{1}{2}x^2 + \frac{3}{5}x^3 + \frac{1}{4}x^4 + ...  
J 1 + x + x^2 + x^3 + x^4 + ...  
K 1 + 2x + x^2 + \frac{3}{5}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5 + ...  
L x^2 - \frac{1}{3}x^3 + \frac{1}{12}x^4 + ...  
M x + \frac{1}{3}x^3 + \frac{1}{12}x^5 + ...  

N 1 - \frac{1}{3}x^2 + \frac{1}{10}x^4 - ...  
O x - \frac{1}{5}x^2 + \frac{1}{6}x^3 - \frac{25}{12}x^4 + ...  
P x^2 - \frac{1}{6}x^4 + \frac{1}{24}x^6 - ...  
Q 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{6}x^6 + ...  
R 1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + ...  
S 1 - \frac{1}{2}x^2 + \frac{1}{23}x^4 + ...  
T 1 - x + x^2 - x^3 + x^4 - ...  
U x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...  
V 1 - \frac{1}{3}x^3 + \frac{1}{5}x^4 - \frac{1}{7}x^5 + ...  
W x + x^3 + x^5 + x^7 + ...  
X 1 + \frac{1}{4}x^2 + \frac{1}{23}x^3 + \frac{1}{50}x^4 + ...  
Y 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{3}{8}x^3 + ...  
Z none of these

3. (1 pt each, 10 pts total) Label each series as either A (Absolutely convergent), or C (Conditionally convergent or D (Divergent). each of the following infinite series converges or diverges.

(i) \( 1 + 1 + 1 + ... \)  
(ii) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)  
(iii) \( 1 - 1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + ... \)  
(iv) \( \sum_{n=1}^{\infty} n^{2n-2} \)  
(v) \( \sum_{n=1}^{\infty} \frac{n}{n^2} \)  
(vi) \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... \)  
(vii) \( \sum_{n=0}^{\infty} \frac{\cos(n)}{n^2} \)  
(viii) \( \sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n} \)  
(ix) \( \sum_{n=1}^{\infty} (\ln n)^{-2} \)  
(x) \( \sum_{n=0}^{\infty} \sin(n) \)
4. (1 pt each, 10 pts total) Evaluate each derivative or find some integral and put the letter of the correct answer in the box.

(i) \[ \frac{d}{dx} \ln x \] \hspace{1cm} (iv) \[ \frac{d}{dx} \cosh^{-1}(x) \] \hspace{1cm} (vii) \[ \int \sec x \, dx \]

(ii) \[ \frac{d}{dx} \arcsin(x) \] \hspace{1cm} (v) \[ \frac{d}{dx} \cosh x \] \hspace{1cm} (viii) \[ \int x \cos x \, dx \]

(iii) \[ \frac{d}{dx} x \sin x \] \hspace{1cm} (vi) \[ \int \frac{dx}{\sqrt{x^2 - 4x^2}} \] \hspace{1cm} (ix) \[ \int \sin^3 x \cos^2 x \, dx \]

(x) \[ \int \frac{dx}{1 + x^2} \]

A \( x \sin x (\frac{1}{x} \sin x + \ln x \cos x) \) 
B \( \frac{1}{\sqrt{1-x^2}} \) 
C \( \frac{1}{x} \) 
D \( -\sinh x \) 
E \( \tan(x) \) 
F \( x \sin x \ln x \cos x \) 
G \( e^x \) 
H \( \ln |\sec x + \tan x| \) 
I \( x \cos x + \sin x \) 
J \( \frac{1}{2} \arcsin(\frac{x}{\sqrt{2}}) \) 
K \( x \ln x \) 
L \( \ln |\cos x + \tan x| \) 
M \( 1/\sqrt{1-x^2} \)

N \( 1/(|x|\sqrt{x^2-1}) \) 
O \( \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x \) 
P \( \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x \) 
Q \( \cos^5 x - \sin^3 x \) 
R \( \cosh x \) 
S \( \cot x \) 
T \( \tanh x \) 
U \( \frac{1}{\sqrt{x^2-1}} \) 
V \( \arcsin(x) \) 
W \( x \cos x \) 
X \( x \sin x + \cos x \) 
Y \( \sinh x \) 
Z none of these

5. (5 pts each, 10 pts total) Do TWO of the following problems (your choice). Put a mark in the box next to the two problems you want to be graded. Put your work on the following pages and clearly mark which problem you are doing.

(i) \[ \text{Solve the following differential equation: } (x + 1) \frac{dy}{dx} - 2(x^2 + x)y = e^x/(x + 1). \]

(ii) \[ \text{State Taylor’s theorem (or the remainder estimation theorem) and use it to prove that the Maclaurin series for } \sinh x \text{ converges to } \sinh x \text{ for all real numbers.} \]

(iii) \[ \text{Find a power series solution (up to and including the } x^4 \text{ term) to the differential equation } y'' - y' - y = 0 \text{ with } y'(0) = 1 \text{ and } y(0) = 1 \]

(iv) \[ \text{Give an example of a series so that } \sum_{n=1}^{\infty} a_n \text{ converges but such that } \sum_{n=1}^{\infty} (a_n)^2 \text{ diverges.} \]

(v) \[ \text{Prove } e = \sum_{n=0}^{\infty} \frac{1}{n!} \text{ is irrational.} \]