PS2-Problem 1 (20pts)

(a; 10pts) Use the second-order integrating factor method to find the real general solution of

\[ y'' + 4y = 4\cos 2t. \]  

(1)

Here is one approach. The general real solution \( y = y(t) \) of this equation is given by \( y = \text{Re} \, z \), where \( z = z(t) \) is the complex general solution of

\[ z'' + 4z = 4e^{2it}. \]  

(2)

The characteristic polynomial for this equation is

\[ \lambda^2 + 0 \cdot \lambda + 4 = (\lambda + 2i)(\lambda - 2i). \]

Thus, the two characteristic roots are \( \lambda_1 = 2i \) and \( \lambda_2 = -2i \), and

\[ (e^{((-2i) - (2i))t}(e^{(-2i)t}z))' = e^{(-2i)t}(z'' + 4z). \]  

(3)

Multiplying both sides of (2) by \( e^{-2it} \) and using (3), we obtain

\[ z'' + 4z = 4e^{2it} \implies e^{-2it}(z'' + 4z) = 4 \implies (e^{-4it}(e^{2it}z))' = 4. \]

Integrating twice, we obtain

\[ (e^{-4it}(e^{2it}z))' = 4 \implies e^{-4it}(e^{2it}z)' = 4t + C_1 \implies (e^{2it}z)' = 4te^{4it} + C_1e^{4it} \implies e^{2it}z = \int (4te^{4it} + C_1e^{4it})dt = \frac{4}{4i}(te^{4it} - \int e^{4it}dt) + C_1e^{4it} \]

\[ = \frac{1}{4}te^{4it} + \frac{1}{4}e^{4it} + \frac{C_1}{4i}e^{4it} + C_2. \]

Since we can replace \((1/4) + (C_1/4i)\) with \( C_1 \), the general solution of (2) is

\[ z(t) = \frac{1}{i}te^{2it} + C_1e^{2it} + C_2e^{-2it}. \]

Taking the real part of this equation and modifying the constants, we obtain

\[ y(t) = \text{Re} \, z(t) = t\sin 2t + C_1\cos 2t + C_2\sin 2t. \]
Here is another approach. The characteristic polynomial and roots for the original equation are the same as for its complex version. Thus, (3) holds with \( z \) replaced by \( y \), and
\[
y'' + 4y = 4\cos 2t \quad \implies \quad e^{-2it}(y'' + 4y) = 4e^{-2it}\cos 2t \quad \implies \quad (e^{-4it}(e^{2it}y))' = 4e^{-2it}\cos 2t.
\]
Integrating the last expression once, we obtain
\[
e^{-4it}(e^{2it}y)' = \int 4e^{-2it}\cos 2t \, dt = 4\int \cos^2 2t \, dt - 4i\int \cos 2t \sin 2t \, dt
\]
\[
= 2\int (\cos 4t + 1) dt - 2i\int \sin 4t \, dt = \frac{1}{2}\sin 2t + 2t + \frac{i}{2}\cos 4t + C_1 = \frac{i}{2}e^{-4it} + 2t + C_1.
\]
The second and last equalities above follow from Euler’s formula, applied in opposite directions. The third inequality uses the half-angle trigonometric formulas. Finally, proceeding as in the second integration step of the first approach, we obtain
\[
e^{2it}y = \int (2te^{4it} + C_1e^{4it} + \frac{i}{2}) \, dt = \frac{1}{2it}te^{4it} + \frac{1}{8}e^{4it} + \frac{C_1}{4t}e^{4it} + \frac{it}{2} + C_2
\]
\[
\implies \quad y(t) = \frac{t}{2i}(e^{2it} - e^{-2it}) + C_1e^{2it} + C_2e^{-2it} = t\sin 2t + C_1e^{2it} + C_2e^{-2it}.
\]
As before, the complex form \( C_1e^{2it} + C_2e^{-2it} \) is equivalent to the real form \( A_1\cos 2t + A_2\sin 2t \).

Remarks: (1) When the nonhomogeneous term, i.e. RHS in (1), is \( \cos \omega t \) or \( \sin \omega t \), the first approach, i.e. complexifying the ODE, is generally faster, but riskier if you are not used to complex numbers. This is the case whether you use the second-order integrating factor approach or the method of undetermined coefficients. Note that if the forcing term is \( \sin \omega t \), you would need to take the imaginary part of the complex solution.

(2) The complex form \( C_1e^{at+ibt} + C_2e^{at-ibt} \) of the general solution of an ODE is always equivalent to the real form \( A_1e^{at}\cos bt + A_2e^{at}\sin bt \).

(b: 10pts) Use the second-order integrating factor method to find the real general solution of
\[
y'' + 5y' + 4y = t \cdot e^{-t}.
\]
In this case, the characteristic polynomial is
\[
\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4).
\]
Thus, the two characteristic roots are \( \lambda_1 = -1 \) and \( \lambda_2 = -4 \), and
\[
(e^{t(-4)-t(-1)})(e^{-4t}y)' = e^{-(-1)t}(y'' + 5y' + 4y).
\]
(5)
Multiplying both sides of (4) by \( e^t \) and using (5), we obtain
\[
y'' + 5y' + 4y = t \cdot e^{-t} \quad \implies \quad e^t(y'' + 5y' + 4y) = t \quad \implies \quad (e^{-3t}(e^{4t}y)') = t.
\]
Integrating twice, we obtain
\[ e^{-3t}(e^{4t}y)' = \int t \, dt = \frac{1}{2} t^2 + C_1 \quad \implies \quad (e^{4t}y)' = \frac{1}{2} t^2 e^{3t} + C_1 e^{3t} \]
\[ \implies \quad e^{4t}y(t) = \frac{1}{2} \int t^2 e^{3t} \, dt + C_1 \int e^{3t} \, dt = \frac{1}{6} (t^2 e^{3t} - \int 2te^{3t} \, dt) + \frac{C_1}{3} e^{3t} \]
\[ = \frac{1}{6} t^2 e^{3t} - \frac{1}{9} (te^{3t} - \int e^{3t} \, dt) + \frac{C_1}{3} e^{3t} = \frac{1}{6} t^2 e^{3t} - \frac{1}{9} te^{3t} + \frac{1}{27} e^{3t} + \frac{C_1}{3} e^{3t} + C_2. \]

Since we can replace $\frac{1}{27} + \frac{C_1}{3}$ with $C_1$, the general solution of (4) is
\[ y(t) = \frac{1}{6} t^2 e^{-t} - \frac{1}{5} te^{-t} + C_1 e^{-t} + C_2 e^{-4t} \]

Remark: In these two cases, the second-order integrating factor approach is not any easier and perhaps a bit harder than the method of undetermined coefficients. In general, the method of undetermined coefficients will be faster whenever it is applicable, i.e. you know what form a solution should have. On the other hand, the integrating factor approach works for all forcing terms.

Section 4.1, Problems 12,14 (18pts)

4.1:12; Spts: Show that $y_1(t) = e^{-t} \cos 2t$ and $y_2(t) = e^{-t} \sin 2t$ form a fundamental set of solutions for
\[ y'' + 2y' + 5y = 0. \]
Find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.

The functions $y_1(t)$ and $y_2(t)$ are linearly independent, since $\tan 2t = y_2(t)/y_1(t)$ is not a constant function. Thus, in order to prove the first statement, we only need to check that $y_1(t)$ and $y_2(t)$ solve the ODE:
\[ y_1'(t) = e^{-t} (-2 \sin 2t - \cos 2t) \quad \implies \quad y_1''(t) = e^{-t} (-4 \cos 2t + 2 \sin 2t + 2 \sin 2t + \cos 2t) \]
\[ = e^{-t} (4 \sin 2t - 3 \cos 2t); \]
\[ y_2'(t) = e^{-t} (2 \cos 2t - \sin 2t) \quad \implies \quad y_2''(t) = e^{-t} (-4 \sin 2t - 2 \cos 2t - 2 \cos 2t + \sin 2t) \]
\[ = -e^{-t} (4 \cos 2t + 3 \sin 2t). \]

Plugging these expressions into the ODE, we obtain
\[
\begin{align*}
y_1'' + 2y'_1 + 5y_1 &= e^{-t} (4 \sin 2t - 3 \cos 2t - 4 \sin 2t - 2 \cos 2t + 5 \cos 2t) = 0; \\
y_1'' + 2y'_1 + 5y_1 &= e^{-t} (-4 \cos 2t - 3 \sin 2t + 4 \cos 2t - 2 \sin 2t + 5 \sin 2t) = 0,
\end{align*}
\]
as needed. For the initial-value problem, we need to find $C_1$ and $C_2$ such that $y(0) = -1$ and $y'(0) = 0$ if $y = C_1 y_1 + C_2 y_2$. Using the above expressions for $y_1'$ and $y_2'$, we find that
\[
y(0) = C_1 = -1 \quad \text{and} \quad y'(0) = -C_1 + 2C_2 = 0.
\]
Thus, \( C_2 = -\frac{1}{2} \), and the solution to the initial value problem is \( y(t) = -e^{-t} \cos 2t - \frac{1}{2}e^{-t} \sin 2t \).

4.1:14 (a; 2pts) **Show that** \( y_1(t) = t^2 \) **is a solution of**

\[
t^2y'' + ty' - 4y = 0. \tag{6}
\]

We need to plug in \( y_1 \) into (6). Since \( y_1' = 2t \) and \( y_1'' = 2 \),
\[
t^2y_1'' + ty_1' - 4y_1 = t^2 \cdot 2 + t \cdot 2t - 4 \cdot t^2 = 0,
\]
as needed.

(b; 8pts) **Let** \( y_2(t) = v(t)y_1(t) = v(t)t^2 \). **Show that** \( y_2 \) **is a solution of (6) if and only if** \( v \) **satisfies**

\[
5v' + tv'' = 0. \tag{7}
\]

**Solve this equation for** \( v \) **and describe the general solution of (6).**

We need to plug in \( y_2 \) into (6):

\[
y_2'(t) = t^2v'(t) + 2tv(t) \implies y_2''(t) = t^2v''(t) + 2tv'(t) + 2v(t) = t^2v'' + 4tv' + 2v
\]
\[
\implies 0 = t^2y_2'' + ty_2 - 4y_2 = (t^4v'' + 4t^3v' + 2t^2) + (t^3v' + 2t^2v) - 4t^2v = t^4v'' + 5t^3v'.
\]

Dividing the last expression by \( t^3 \), we obtain (7). In order to solve (7), we first divide this equation by \( t \) and then multiply by the integrating factor \( e^{\int{\frac{5}{t}}dt} = |t|^5 \), or just by \( t^5 \):

\[
5v' + tv'' = 0 \implies t^5v'' + 5t^4v' = 0 \implies (t^5v')' = 0 \implies t^5v'(t) = C_1
\]
\[
\implies v'(t) = C_1t^{-5} \implies v(t) = -\frac{C_1}{4}t^{-4} + C_2.
\]

Since we need to find a single non-constant solution of (7), we can take
\[
v(t) = t^{-4} \quad \text{and} \quad y_2(t) = v(t)y_1(t) = t^{-4}t^2 = t^{-2}.
\]

The general solution of (6) is thus given \( y(t) = C_1t^2 + C_2t^{-2} \).

**Section 4.2, Problems 4 (4pts)**

**Use the substitution** \( v = y' \) **to write the second-order ODE**

\[
y'' + 2y' + 2y = \sin 2\pi t
\]
**as a system of two first-order equations.**

Since \( v = y' \),
\[
v' = y'' = -2y' - 2y + \sin 2\pi t = -2v - 2y + \sin 2\pi t.
\]
Thus, the above second-order ODE is equivalent to the system
\[
\begin{cases}
y' = v \\
v' = -2v - 2y + \sin 2\pi t.
\end{cases}
\]
Section 4.3, Problems 4,10,14,26 (26pts)

4.3:4; 5pts: Find the general solution of the ODE
\[ 2y'' - y' - y = 0. \]
The characteristic polynomial for this equation is
\[ 2\lambda^2 - \lambda - 1 = (2\lambda + 1)(\lambda - 1). \]
Thus, the two characteristic roots are \( \lambda_1 = -1/2 \) and \( \lambda_2 = 1 \). Since they are real and distinct, and the general solution of the ODE is
\[ y(t) = C_1 e^t + C_2 e^{-t/2}. \]

4.3:10; 8pts: Find the general solution of the ODE
\[ y'' + 2y' + 17y = 0. \]
The characteristic polynomial for this equation is
\[ \lambda^2 + 2\lambda + 17 = (\lambda - \lambda_1)(\lambda - \lambda_2), \quad \lambda_1, \lambda_2 = -1 \pm \sqrt{17} = -1 \pm 4i. \]
Thus, the two characteristic roots are complex, and so is the general solution of the ODE
\[ y(t) = C_1 e^{(-1+4i)t} + C_2 e^{(-1-4i)t}. \]
The corresponding general real solution is given by
\[ y(t) = C_1 e^{-t} \cos 4t + C_2 e^{-t} \sin 4t. \]

4.3:14; 5pts: Find the general solution of the ODE
\[ y'' - 6y' + 9y = 0. \]
The characteristic polynomial for this equation is
\[ \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2. \]
Thus, this equation has a repeated root, \( \lambda = 3 \), and the general solution of the ODE is
\[ y(t) = C_1 e^{3t} + C_2 te^{3t}. \]

4.3:26; 8pts: Find the solution to the initial value problem
\[ 4y'' + y = 0, \quad y(1) = 0, \quad y'(1) = -2. \]
The characteristic polynomial for this equation is
\[ 4\lambda^2 + 1 = (2\lambda + i)(2\lambda - i). \]
Thus, the two roots, \( \lambda_1 = i/2 \) and \( \lambda = -i/2 \) are distinct, and the general (complex) solution is
\[
y(t) = C_1 e^{it/2} + C_2 e^{-it/2}.
\]

The initial conditions \( y(1) = 0 \) and \( y'(1) = -2 \) give
\[
0 = y(1) = C_1 e^{i/2} + C_2 e^{-i/2} \quad \text{and} \quad -2 = y'(1) = C_1 \frac{i}{2} e^{i/2} - C_2 \frac{i}{2} e^{-i/2}.
\]

Thus, \( C_1 = 2ie^{-i/2} \) and \( C_2 = -2ie^{i/2} \), and
\[
y(t) = 2ie^{-i/2} e^{it/2} - 2ie^{i/2} e^{-it/2} = 2i(e^{i(t-1)/2} - e^{-i(t-1/2)})
\]
\[
= 2i \cdot 2i \sin((t-1)/2) = -4 \sin((t-1)/2).
\]

Thus, the solution to the initial value problem is \( y(t) = -4 \sin((t-1)/2) \) Please check that this function indeed satisfies the ODE and the initial conditions.

**Section 4.4, Problem 17 (8pts)**

Prove that an overdamped solution of \( m\ddot{y} + \mu \dot{y} + ky = 0 \) can cross the time axis no more than once.

Rewrite the given equation as
\[
y'' + \frac{\mu}{m} y' + \frac{k}{m} = 0 \implies y'' + 2cy' + \omega_0^2 y = 0,
\]
where \( 2c = \mu/m \) and \( \omega_0^2 = k/m \). The characteristic equation is \( \lambda^2 + 2c\lambda + \omega_0^2 = 0 \). Its roots are
\[
\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and} \quad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}
\]

Since the system is overdamped, \( c^2 - \omega_0^2 > 0 \), and we have two distinct real roots \( \lambda_1 \neq \lambda_2 < 0 \). The general solution is of the form
\[
y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.
\]

The number of times any such curve crosses the \( t \)-axis is the number of values of \( t \) for which
\[
C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = e^{\lambda_1 t}(C_1 + C_2 e^{(\lambda_2 - \lambda_1) t}) = 0.
\]

Since \( e^{\lambda_1 t} \) is never zero, the point \( (t, y(t)) \) will lie on the \( t \)-axis if and only if
\[
C_1 + C_2 e^{(\lambda_2 - \lambda_1) t} = 0 \implies e^{(\lambda_2 - \lambda_1) t} = -\frac{C_1}{C_2}
\]

Now, if \( C_1/C_2 \geq 0 \), the right hand side is negative or zero. It has no logarithm and hence there are no times \( t \) where \( y(t) = 0 \). If \( C_1/C_2 < 0 \), the solution curve intersects the \( t \)-axis only at time
\[
t = \frac{1}{\lambda_2 - \lambda_1} \ln \left( -\frac{C_1}{C_2} \right)
\]

Note that \( \lambda_1 \neq \lambda_2 \) above. Thus, the solution curve never intersects the \( t \)-axis more than once.
Section 4.5, Problems 2, 6, 16, 18, 26, 30, 32, 42 (74pts)

4.5:2; 6pts: Using an exponential forcing term, find a particular solution of the equation
\[ y'' + 6y' + 8y = -3e^{-t}. \]
We look for the particular solution of the form \( y_p(t) = Ae^{-t}. \) After making the substitutions:
\[ y_p(t) = A^{-t}, \quad y_p'(t) = -Ae^{-t}, \quad y_p''(t) = Ae^{-t}, \]
the equation becomes:
\[ Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t} \implies 3Ae^{-t} = -3e^{-t} \implies A = -1. \]
Thus, a particular solution is \( y(t) = -e^{-t}. \)

4.5:6; 8pts: Use the form \( y = a \cos \omega t + b \sin \omega t \) to find a particular solution of the equation
\[ y'' + 9y = \sin 2t \]
Let \( y_p(t) = a \cos 2t + b \sin 2t. \) After making the substitutions:
\[ y_p(t) = a \cos 2t + b \sin 2t, \quad y_p'(t) = -2a \sin 2t + 2b \cos 2t, \quad y_p''(t) = -4a \cos 2t - 4b \sin 2t, \]
the equation \( y'' + 9y = \sin 2t \) becomes:
\[ -4a \cos 2t - 4b \sin 2t + 9a \cos 2t + 9b \sin 2t = \sin 2t \]
\[ \implies 5a \cos 2t + 5b \sin 2t = \sin 2t \implies a = 0, \quad b = \frac{1}{5} \]
A particular solution is \( y(t) = \frac{1}{5} \sin 2t. \)

4.5:16; 8pts: Find a particular solution of the equation
\[ y'' + 5y' + 6y = 4 - t^2 \]
The forcing term is a quadratic polynomial, so we look for a particular solution of the form
\[ y_p(t) = at^2 + bt + c, \quad \implies y_p'(t) = 2at + b, \quad \implies y_p''(t) = 2a. \]
The equation becomes:
\[ y'' + 5y' + 6y = 4 - t^2 \implies 2a + 5(2at + b) + 6(at^2 + bt + c) = 4 - t^2 \]
\[ \implies 6at^2 + (10a + 6b)t + (2a + 5b + 6c) = -t^2 + 4. \]
Thus, \( a, b, c \) must satisfy:
\[ 6a = -1, \quad 10a + 6b = 0, \quad 2a + 5b + 6c = 4 \implies a = -\frac{1}{6}, \quad b = \frac{5}{18}, \quad c = \frac{53}{108}. \]
So, a particular solution is $y_p(t) = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}$.

4.5:18; 12pts: For the equation $y'' + 3y' + 2y = 3e^{-4t}$, first solve the associated homogeneous equation, then find a particular solution. Using Theorem 5.2, form the general solution, and then find the solution satisfying initial conditions $y(0) = 1$, $y'(0) = 0$.

The homogeneous equation $y'' + 3y' + 2y = 0$ has characteristic equation

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0,$$

with zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, the homogeneous solution is

$$y_h(t) = C_1e^{-t} + C_2e^{-2t}.$$

For $y_p = Ae^{-4t}$, $y'_p = -4Ae^{-4t}$ and $y''_p = 16Ae^{-4t}$. Substituting into the inhomogeneous ODE, we get

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t} \implies 6A = 3 \implies A = \frac{1}{2}.$$

Thus, a particular solution is $y_p(t) = \frac{1}{2}e^{-4t}$. By Theorem 5.2, the general solution is

$$y = C_1e^{-t} + C_2e^{-2t} + \frac{1}{2}e^{-4t}.$$

The given initial conditions imply:

$$y(0) = C_1 + C_2 + \frac{1}{2} = 1, \quad y'(0) = -C_1 - 2C_2 - 2 = 0 \implies C_1 = 3, \ C_2 = -\frac{5}{2}.$$

So, the solution to the initial value problem is $y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}$.

4.5:26; 10pts: In the equation $y'' + 4y = 4\cos 2t$, the forcing term is also a solution of the associated homogeneous equation. Use this to find a particular solution.

Our strategy is to look at the equation $z'' + 4z = e^{2it}$, of which the given equation is the real part. The characteristic equation of the homogeneous equation $z'' + 4z = 0$ is $\lambda^2 + 4 = 0$. Its roots are ±2i. So, the homogeneous solution is:

$$z_h = C_1e^{2it} + C_2e^{-2it}.$$

The forcing term of $z'' + 4z = 4e^{2it}$ is also a solution of the homogeneous equation. Thus, we try to find a particular solution of the form $z_p = Ate^{2it}$:

$$z_p = Ate^{2it} \implies z'_p = Ae^{2it}(1 + 2it) \implies z''_p = 4Ae^{2it}(i - t).$$

After substituting these into $z'' + 4z = 4e^{2it}$, we get:

$$4Ae^{2it}(i - t) + 4Ate^{2it} = 4e^{2it} \implies 4iA = 4 \implies A = \frac{1}{i} = -i \implies z_p = -ite^{2it} = -it(\cos 2t + i\sin 2t) = t\sin 2t - it\cos 2t.$$
Its real part is a particular solution we are looking for: $y_p = \text{Re}(z_p) = t \sin 2t$

**4.5:30; 10pts:** If $y_f(t)$ and $y_g(t)$ are solutions of

$$y'' + py' + qy = f(t) \quad \text{and} \quad y'' + py' + qy = g(t),$$

respectively, show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t),$$

where $\alpha$ and $\beta$ are any real numbers.

We are given that:

$$y'' + py' + qy = f(t) \quad \text{and} \quad y'' + py' + qy = g(t)$$

We plug in $z(t)$ into $y'' + py' + qy = \alpha f(t) + \beta g(t)$ and use these two properties of $y_f$ and $y_g$:

$$z'' + pz' + qz = (\alpha y_f + \beta y_g)'' + p(\alpha y_f + \beta y_g)' + q(\alpha y_f + \beta y_g)$$

$$= (\alpha y_f'' + \beta y_g'') + p(\alpha y_f' + \beta y_g') + q(\alpha y_f + \beta y_g)$$

$$= \alpha(y_f'' + py_f' + qy_f) + \beta(y_g'' + py_g' + qy_g)$$

$$= \alpha f(t) + \beta g(t).$$

Thus, $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of $y'' + py' + qy = \alpha f(t) + \beta g(t)$.

**4.5:32; 12pts:** Use the previous exercise to find a particular solution of the equation

$$y'' - y = t - e^{-t}.$$

The forcing term is the linear combination $t - e^{-t} = 1 \cdot t + (-1)e^{-t}$. We first find a particular solution $y_{p_1}$ of $y'' - y = t$, and then a particular solution $y_{p_2}$ of $y'' - y = -e^{-t}$. By the previous exercise, $y_{p_1} - y_{p_2}$ will be a particular solution to our equation. To find $y_{p_1}$, substitute $y = at + b$ into

$$y'' - y = t \implies -at - b = t \implies a = -1, \ b = 0, \implies y_{p_1}(t) = -t.$$

To find $y_{p_2}$, note that the characteristic equation for the homogeneous equation $y'' - y = 0$ is $\lambda^2 - 1 = 0$. Its roots are $\lambda_1 = -1$ and $\lambda_2 = 1$, giving the homogeneous solution

$$y_h = C_1 e^{-t} + C_2 e^t.$$

It follows that the forcing term $e^{-t}$ is a solution of the homogeneous equation. So we try to find $y_{p_2}$ of the form $y_{p_2}(t) = Ate^{-t}$:

$$y_{p_2} = Ate^{-t} \implies y_{p_2}' = Ae^{-t}(1 - t) \implies y_{p_2}'' = Ae^{-t}(t - 2).$$

The equation now becomes:

$$e^{-t} = y_{p_2}'' - y_{p_2} = Ae^{-t}(t - 2) - Ate^{-t} \implies -2A = 1 \implies A = -\frac{1}{2} \implies y_{p_2}(t) = -\frac{1}{2}te^{-t}.$$
For the first part, plug in $y = t - e^{-t}$ is

$$y_p = y_{p_1} - y_{p_2} = -t + \frac{3}{2}te^{-t}$$

4.5:42: 12pts: Find a particular solution of the equation $y'' + 5y' + 4y = te^{-t}$.

The characteristic equation for the corresponding homogeneous equation $y'' + 5y' + 4y = 0$ is

$$\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0.$$ 

Its are roots $\lambda_1 = -1$ and $\lambda_2 = -4$, and the homogeneous solution is

$$y_h = C_1 e^{-t} + C_2 e^{-t}.$$ 

In particular, $e^{-t}$ is a solution to the homogeneous equation. Thus, we modify the hint in Exercise 39, and look for a particular solution of the form $y_p = t(at + b)e^{-t}$:

$$y_p(t) = t(at + b)e^{-t} \quad \implies \quad y_p'(t) = (-at^2 + (2a - b)t + b)e^{-t} \quad \implies \quad y_p''(t) = (at^2 + (-4a + b)t + (2a - 2b))e^{-t}$$

Substituting, we get:

$$te^{-t} = y'' + 5y' + 4y = (6at + (2a + 3b))e^{-t} \implies 6a = 1, 2a + 3b = 0, \quad a = \frac{1}{6}, \quad b = -\frac{1}{9}.$$ 

Thus, a solution of $y'' + 5y' + 4y = te^{-t}$ is

$$y_p = \frac{1}{6}t^2e^{-t} - \frac{1}{9}te^{-t}$$

Section 4.6, Problem 13

Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0.$$ 

Use variation of parameters to find the general solution to

$$t^2y'' + 3ty' - 3y = t^{-1}.$$ 

For the first part, plug in $y_1(t) = t$ and $y_2(t) = t^{-3}$ into the homogeneous equation:

$$y_1 = t, \quad y_1' = 1, \quad y_1'' = 0 \implies t^2y'' + 3ty' - 3y = t^2 \cdot 0 + 3t \cdot 1 - 3 \cdot t = 0;$$

$$y_1 = t^{-3}, \quad y_1' = -3t^{-4}, \quad y_1'' = 12t^{-5} \implies t^2y'' + 3ty' - 3y = t^2 \cdot 0 + 3t \cdot 0 - 3 \cdot t^{-3} = 0,$$

as needed. We look for a solution to the inhomogeneous equation of the form $y_p = v_1y_1 + v_2y_2$. Then,

$$y'_p = (v_1' y_1 + y_1v_2') + y_1' v_1 + y_2' v_2 = (tv_1' + t^{-3}v_2') + v_1 - 3t^{-4}v_2.$$ 

We set the expression in the parenthesis to zero. Thus,

$$y'_p = v_1 - 3t^{-4}v_2 \implies y''_p = v_1' + 12t^{-5}v_2 - 3t^{-4}v_2' = t^2y'' + 3ty' - 3y = t^2v_1' - 3t^{-2}v_2' = t^{-1}.$$ 

Since we also assumed that $tv_1' + t^{-3}v_2' = 0$, we need to solve the system

\[
\begin{cases} 
  v_1' + t^{-4}v_2' = 0 \\
  v_1' - 3t^{-4}v_2' = t^{-3} 
\end{cases}
\]

$$\implies v_1' = \frac{1}{4}t^{-3}, \quad v_2' = -\frac{1}{4}t \implies v_1 = -\frac{1}{8}t^{-2}, \quad v_2 = -\frac{1}{8}t^2$$

$$\implies y_p = v_1y_1 + v_2y_2 = -\frac{1}{8}t^{-2} \cdot t - \frac{1}{8}t^2 \cdot t^{-3} = -\frac{1}{4}t^{-1}.$$ 

Thus, the general solution is

$$y(t) = C_1t + C_2t^{-3} - \frac{1}{4}t^{-1}$$