Math53: Ordinary Differential Equations
Winter 2004

Solutions to Problem Set 2

PS2-Problem 1 (20pts)

(a; **10pts**) *Use the second-order integrating factor method to find the real general solution of*

$$y'' + 4y = 4\cos 2t. \tag{1}$$

Here is one approach. The general real solution $y = y(t)$ of this equation is given by $y = \operatorname{Re} z$, where $z = z(t)$ is the complex general solution of

$$z'' + 4z = 4e^{2it}. \tag{2}$$

The characteristic polynomial for this equation is

$$\lambda^2 + 0\cdot\lambda + 4 = (\lambda + 2i)(\lambda - 2i).$$

Thus, the two characteristic roots are $\lambda_1 = 2i$ and $\lambda_2 = -2i$, and

$$\big(e^{((-2i)-(2i))t}(e^{-(-2i)t}z)'\big)' = e^{-(2i)t}(z'' + 4z). \tag{3}$$

Multiplying both sides of (2) by e^{-2it} and using (3), we obtain

$$z'' + 4z = 4e^{2it} \quad\Longrightarrow\quad e^{-2it}(z'' + 4z) = 4 \quad\Longrightarrow\quad \big(e^{-4it}(e^{2it}z)'\big)' = 4.$$

Integrating twice, we obtain

$$\begin{aligned}
\big(e^{-4it}(e^{2it}z)'\big)' = 4 \quad&\Longrightarrow\quad e^{-4it}(e^{2it}z)' = 4t + C_1 \quad\Longrightarrow\quad (e^{2it}z)' = 4te^{4it} + C_1e^{4it} \\
&\Longrightarrow\quad e^{2it}z = \int(4te^{4it} + C_1e^{4it})dt = \frac{4}{4i}\big(te^{4it} - \int e^{4it}dt\big) + \frac{C_1}{4i}e^{4it} \\
&\qquad\qquad = \frac{1}{i}te^{4it} + \frac{1}{4}e^{4it} + \frac{C_1}{4i}e^{4it} + C_2.
\end{aligned}$$

Since we can replace $(1/4)+(C_1/4i)$ with C_1, the general solution of (2) is

$$z(t) = \frac{1}{i}te^{2it} + C_1e^{2it} + C_2e^{-2it}.$$

Taking the real part of this equation and modifying the constants, we obtain

$$\boxed{y(t) = \operatorname{Re} z(t) = t\sin 2t + C_1\cos 2t + C_2\sin 2t}$$

Here is another approach. The characteristic polynomial and roots for the original equation are the same as for its complex version. Thus, (3) holds with z replaced by y, and

$$y''+4y=4\cos 2t \quad\Longrightarrow\quad e^{-2it}(y''+4y)=4e^{-2it}\cos 2t \quad\Longrightarrow\quad \big(e^{-4it}(e^{2it}y)'\big)'=4e^{-2it}\cos 2t.$$

Integrating the last expression once, we obtain

$$\begin{aligned}e^{-4it}(e^{2it}y)' &= \int 4e^{-2it}\cos 2t\,dt = 4\int\cos^2 2t\,dt-4i\int\cos 2t\,\sin 2t\,dt\\ &= 2\int(\cos 4t+1)dt-2i\int\sin 4t\,dt=\frac{1}{2}\sin 2t+2t+\frac{i}{2}\cos 4t+C_1=\frac{i}{2}e^{-4it}+2t+C_1.\end{aligned}$$

The second and last equalities above follow from Euler's formula, applied in opposite directions. The third inequality uses the half-angle trigonometric formulas. Finally, proceeding as in the second integration step of the first approach, we obtain

$$\begin{aligned}e^{2it}y &= \int\Big(2te^{4it}+C_1e^{4it}+\frac{i}{2}\Big)dt=\frac{1}{2i}te^{4it}+\frac{1}{8}e^{4it}+\frac{C_1}{4i}e^{4it}+\frac{it}{2}+C_2\\ \Longrightarrow\quad y(t) &= \frac{t}{2i}\big(e^{2it}-e^{-2it}\big)+C_1e^{2it}+C_2e^{-2it}=t\sin 2t+C_1e^{2it}+C_2e^{-2it}.\end{aligned}$$

As before, the complex form $C_1e^{2it}+C_2e^{-2it}$ is equivalent to the real form $A_1\cos 2t+A_2\sin 2t$.

Remarks: (1) When the nonhomogeneous term, i.e. RHS in (1), is $\cos\omega t$ or $\sin\omega t$, the first approach, i.e. complexifying the ODE, is generally faster, but riskier if you are not used to complex numbers. This is the case whether you use the second-order integrating factor approach or the method of undetermined coefficients. Note that if the the forcing term is $\sin\omega t$, you would need to take the imaginary part of the complex solution.
(2) The complex form $C_1e^{at+ibt}+C_2e^{at-ibt}$ of the general solution of an ODE is always equivalent to the real form $A_1e^{at}\cos bt+A_2e^{at}\sin bt$.

(b; **10pts**) *Use the second-order integrating factor method to find the real general solution of*

$$y''+5y'+4y=t\cdot e^{-t}. \tag{4}$$

In this case, the characteristic polynomial is

$$\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4).$$

Thus, the two characteristic roots are $\lambda_1=-1$ and $\lambda_2=-4$, and

$$\big(e^{((-4)-(-1))t}(e^{-(-4)t}y)'\big)'=e^{-(-1)t}(y''+5y'+4y). \tag{5}$$

Multiplying both sides of (4) by e^t and using (5), we obtain

$$y''+5y'+4y=t\cdot e^{-t} \quad\Longrightarrow\quad e^t(y''+5y'+4y)=t \quad\Longrightarrow\quad \big(e^{-3t}(e^{4t}y)'\big)'=t.$$

Integrating twice, we obtain

$$\begin{aligned} e^{-3t}(e^{4t}y)' &= \int t\,dt = \frac{1}{2}t^2 + C_1 \quad \Longrightarrow \quad (e^{4t}y)' = \frac{1}{2}t^2e^{3t} + C_1e^{3t} \\ \Longrightarrow \qquad e^{4t}y(t) &= \frac{1}{2}\int t^2e^{3t}dt + C_1\int e^{3t}dt = \frac{1}{6}\big(t^2e^{3t} - \int 2te^{3t}dt\big) + \frac{C_1}{3}e^{3t} \\ &= \frac{1}{6}t^2e^{3t} - \frac{1}{9}\big(te^{3t} - \int e^{3t}dt\big) + \frac{C_1}{3}e^{3t} = \frac{1}{6}t^2e^{3t} - \frac{1}{9}te^{3t} + \frac{1}{27}e^{3t} + \frac{C_1}{3}e^{3t} + C_2. \end{aligned}$$

Since we can replace $(1/27)+(C_1/3)$ with C_1, the general solution of (4) is

$$\boxed{y(t) = \tfrac{1}{6}t^2e^{-t} - \tfrac{1}{9}te^{-t} + C_1e^{-t} + C_2e^{-4t}}$$

Remark: In these two cases, the second-order integrating factor approach is not any easier and perhaps a bit harder than the method of undetermined coefficients. In general, the method of undetermined coefficients will be faster whenever it is applicable, i.e. you know what form a solution should have. On the other hand, the integrating factor approach works for all forcing terms.

Section 4.1, Problems 12,14 (18pts)

4.1:12; 8pts: *Show that $y_1(t)=e^{-t}\cos 2t$ and $y_2(t)=e^{-t}\sin 2t$ form a fundamental set of solutions for*

$$y'' + 2y' + 5y = 0.$$

Find a solution satisfying $y(0)=-1$ and $y'(0)=0$.
The functions $y_1(t)$ and $y_2(t)$ are linearly independent, since $\tan 2t = y_2(t)/y_1(t)$ is not a constant function. Thus, in order to prove the first statement, we only need to check that $y_1(t)$ and $y_2(t)$ solve the ODE:

$$\begin{aligned} y_1'(t) &= e^{-t}\big(-2\sin 2t - \cos 2t\big) \quad \Longrightarrow \quad y_1''(t) = e^{-t}\big(-4\cos 2t + 2\sin 2t + 2\sin 2t + \cos 2t\big) \\ &\qquad\qquad = e^{-t}\big(4\sin 2t - 3\cos 2t\big); \\ y_2'(t) &= e^{-t}\big(2\cos 2t - \sin 2t\big) \quad \Longrightarrow \quad y_2''(t) = e^{-t}\big(-4\sin 2t - 2\cos 2t - 2\cos 2t + \sin 2t\big) \\ &\qquad\qquad = -e^{-t}\big(4\cos 2t + 3\sin 2t\big). \end{aligned}$$

Plugging these expressions into the ODE, we obtain

$$\begin{aligned} y_1'' + 2y_1' + 5y_1 &= e^{-t}\big(4\sin 2t - 3\cos 2t - 4\sin 2t - 2\cos 2t + 5\cos 2t\big) = 0; \\ y_1'' + 2y_1' + 5y_1 &= e^{-t}\big(-4\cos 2t - 3\sin 2t + 4\cos 2t - 2\sin 2t + 5\sin 2t\big) = 0, \end{aligned}$$

as needed. For the initial-value problem, we need to find C_1 and C_2 such that $y(0) = -1$ and $y'(0)=0$ if $y=C_1y_1+C_2y_2$. Using the above expressions for y_1' and y_2', we find that

$$y(0) = C_1 = -1 \qquad \text{and} \qquad y'(0) = -C_1 + 2C_2 = 0.$$

Thus, $C_2 = -1/2$, and the solution to the initial value problem is $\boxed{y(t) = -e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t}$

4.1:14 (a; **2pts**) *Show that $y_1(t) = t^2$ is a solution of*

$$t^2 y'' + ty' - 4y = 0. \tag{6}$$

We need to plug in y_1 into (6). Since $y_1' = 2t$ and $y_1'' = 2$,

$$t^2 y_1'' + ty_1' - 4y_1 = t^2 \cdot 2 + t \cdot 2t - 4 \cdot t^2 = 0,$$

as needed.
(b; **8pts**) *Let $y_2(t) = v(t)y_1(t) = v(t)t^2$. Show that y_2 is a solution of (6) if and only if v satisfies*

$$5v' + tv'' = 0. \tag{7}$$

Solve this equation for v and describe the general solution of (6).
We need to plug in y_2 into (6):

$$y_2'(t) = t^2 v'(t) + 2tv(t) \implies y_2''(t) = t^2 v''(t) + 2tv'(t) + 2tv'(t) + 2v(t) = t^2 v'' + 4tv' + 2v$$
$$\implies 0 = t^2 y_2'' + ty_2' - 4y_2 = \big(t^4 v'' + 4t^3 v' + 2t^2 v\big) + \big(t^3 v' + 2t^2 v\big) - 4t^2 v = t^4 v'' + 5t^3 v'.$$

Dividing the last expression by t^3, we obtain (7). In order to solve (7), we first divide this equation by t and then multiply by the integrating factor $e^{\int (5/t)dt} = |t|^5$, or just by t^5:

$$5v' + tv'' = 0 \implies t^5 v'' + 5t^4 v' = 0 \implies (t^5 v')' = 0 \implies t^5 v'(t) = C_1$$
$$\implies v'(t) = C_1 t^{-5} \implies v(t) = -\frac{C_1}{4} t^{-4} + C_2.$$

Since we need to find a single non-constant solution of (7), we can take

$$v(t) = t^{-4} \qquad \text{and} \qquad y_2(t) = v(t)y_1(t) = t^{-4}t^2 = t^{-2}.$$

The general solution of (6) is thus given $\boxed{y(t) = C_1 t^2 + C_2 t^{-2}}$

Section 4.2, Problems 4 (4pts)

Use the substitution $v = y'$ to write the second-order ODE

$$y'' + 2y' + 2y = \sin 2\pi t$$

as a system of two first-order equations.
Since $v = y'$,

$$v' = y'' = -2y' - 2y + \sin 2\pi t = -2v - 2y + \sin 2\pi t.$$

Thus, the above second-order ODE is equivalent to the system

$$\begin{cases} y' = v \\ v' = -2v - 2y + \sin 2\pi t. \end{cases}$$

Section 4.3, Problems 4,10,14,26 (26pts)

4.3:4; 5pts: *Find the general solution of the ODE*

$$2y'' - y' - y = 0.$$

The characteristic polynomial for this equation is

$$2\lambda^2 - \lambda - 1 = (2\lambda + 1)(\lambda - 1).$$

Thus, the two characteristic roots are $\lambda_1 = -1/2$ and $\lambda_2 = 1$. Since they are real and distinct, and the general solution of the ODE is $\boxed{y(t) = C_1 e^t + C_2 e^{-t/2}}$

4.3:10; 8pts: *Find the general solution of the ODE*

$$y'' + 2y' + 17y = 0.$$

The characteristic polynomial for this equation is

$$\lambda^2 + 2\lambda + 17 = (\lambda - \lambda_1)(\lambda - \lambda_2), \qquad \lambda_1, \lambda_2 = -1 \pm \sqrt{1-17} = -1 \pm 4i.$$

Thus, the two characteristic roots are complex, and so is the general solution of the ODE

$$y(t) = C_1 e^{(-1+4i)t} + C_2 e^{(-1-4i)t}.$$

The corresponding general real solution is given by $\boxed{y(t) = C_1 e^{-t} \cos 4t + C_2 e^{-t} \sin 4t}$

4.3:14; 5pts: *Find the general solution of the ODE*

$$y'' - 6y' + 9y = 0.$$

The characteristic polynomial for this equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus, this equation has a repeated root, $\lambda = 3$, and the general solution of the ODE is

$$\boxed{y(t) = C_1 e^{3t} + C_2 t e^{3t}}$$

4.3:26; 8pts: *Find the solution to the initial value problem*

$$4y'' + y = 0, \qquad y(1) = 0, \quad y'(1) = -2.$$

The characteristic polynomial for this equation is

$$4\lambda^2 + 1 = (2\lambda + i)(2\lambda - i).$$

Thus, the two roots, $\lambda_1 = i/2$ and $\lambda = -i/2$ are distinct, and the general (complex) solution is

$$y(t) = C_1 e^{it/2} + C_2 e^{-it/2}.$$

The initial conditions $y(1) = 0$ and $y'(1) = -2$ give

$$0 = y(1) = C_1 e^{i/2} + C_2 e^{-i/2} \qquad \text{and} \qquad -2 = y'(1) = C_1 \frac{i}{2} e^{i/2} - C_2 \frac{i}{2} e^{-i/2}.$$

Thus, $C_1 = 2ie^{-i/2}$ and $C_2 = -2ie^{i/2}$, and

$$\begin{aligned} y(t) &= 2ie^{-i/2} e^{it/2} - 2ie^{i/2} e^{-it/2} = 2i\big(e^{i(t-1)/2} - e^{-i(t-1)/2}\big) \\ &= 2i \cdot 2i \sin((t-1)/2) = -4\sin((t-1)/2). \end{aligned}$$

Thus, the solution to the initial value problem is $\boxed{y(t) = -4\sin((t-1)/2)}$ *Please check that this function indeed satisfies the ODE and the initial conditions.*

Section 4.4, Problem 17 (8pts)

Prove that an overdamped solution of $my'' + \mu y k y = 0$ can cross the time axis no more than once. Rewrite the given equation as

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} = 0 \quad \Longrightarrow \quad y'' + 2cy' + \omega_0^2 y = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$. Its roots are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \qquad \text{and} \qquad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}$$

Since the system is overdamped, $c^2 - \omega_0^2 > 0$, and we have two distinct real roots $\lambda_1 \neq \lambda_2 < 0$. The general solution is of the form

$$y(t) = C_1 e^{\lambda_1 t} + C_2^{\lambda_2 t}.$$

The number of times any such curve crosses the t-axis is the number of values of t for which

$$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = e^{\lambda_1 t}(C_1 + C_2 e^{(\lambda_2 - \lambda_1)t}) = 0.$$

Since $e^{\lambda_1 t}$ is never zero, the point $(t, y(t))$ will lie on the t-axis if and only if

$$C_1 + C_2 e^{(\lambda_2 - \lambda_1)t} = 0 \quad \Longrightarrow \quad e^{(\lambda_2 - \lambda_1)t} = -\frac{C_1}{C_2}$$

Now, if $C_1/C_2 \geq 0$, the right hand side is negative or zero. It has no logarithm and hence there are no times t where $y(t) = 0$. If $C_1/C_2 < 0$, the solution curve intersects the t-axis only at time

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(-\frac{C_1}{C_2}\right)$$

Note that $\lambda_1 \neq \lambda_2$ above. Thus, the solution curve never intersects the t-axis more than once.

Section 4.5, Problems 2,6,16,18,26,30,32,42 (74pts)

4.5:2; 6pts: *Using an exponential forcing term, find a particular solution of the equation*

$$y'' + 6y' + 8y = -3e^{-t}.$$

We look for the particular solution of the form $y_p(t) = Ae^{-t}$. After making the substitutions:

$$y_p(t) = A^{-t}, \qquad y_p'(t) = -Ae^{-t}, \qquad y_p''(t) = Ae^{-t},$$

the equation becomes:

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t} \implies 3Ae^{-t} = -3e^{-t} \implies A = -1.$$

Thus, a particular solution is $\boxed{y(t) = -e^{-t}}$

4.5:6; 8pts: *Use the form $y = a\cos\omega t + b\sin\omega t$ to find a particular solution of the equation*

$$y'' + 9y = \sin 2t$$

Let $y_p(t) = a\cos 2t + b\sin 2t$. After making the substitutions:

$$y_p(t) = a\cos 2t + b\sin 2t, \qquad y_p'(t) = -2a\sin 2t + 2b\cos 2t, \qquad y_p''(t) = -4a\cos 2t - 4b\sin 2t,$$

the equation $y'' + 9y = \sin 2t$ becomes:

$$\begin{aligned} &-4a\cos 2t - 4b\sin 2t + 9a\cos 2t + 9b\sin 2t = \sin 2t \\ \implies\ & 5a\cos 2t + 5b\sin 2t = \sin 2t \implies a = 0,\ b = \frac{1}{5} \end{aligned}$$

A particular solution is $\boxed{y(t) = \frac{1}{5}\sin 2t}$

4.5:16; 8pts: *Find a particular solution of the equation*

$$y'' + 5y' + 6y = 4 - t^2$$

The forcing term is a quadratic polynomial, so we look for a particular solution of the form

$$y_p(t) = at^2 + bt + c, \implies y_p'(t) = 2at + b, \implies y_p''(t) = 2a.$$

The equation becomes:

$$\begin{aligned} y'' + 5y' + 6y = 4 - t^2 &\implies 2a + 5(2at + b) + 6(at^2 + bt + c) = 4 - t^2 \\ &\implies 6at^2 + (10a + 6b)t + (2a + 5b + 6c) = -t^2 + 4. \end{aligned}$$

Thus, a, b, c must satisfy:

$$6a = -1, \quad 10a + 6b = 0, \quad 2a + 5b + 6c = 4 \implies a = -\frac{1}{6},\ b = \frac{5}{18},\ c = \frac{53}{108}.$$

So, a particular solution is $\boxed{y_p(t) = -\frac{1}{6}t^2 + \frac{5}{18}t + \frac{53}{108}}$

4.5:18; 12pts: *For the equation*

$$y'' + 3y' + 2y = 3e^{-4t},$$

first solve the associated homogeneous equation, then find a particular solution. Using Theorem 5.2, form the general solution, and then find the solution satisfying initial conditions $y(0)=1$, $y'(0)=0$.
The homogeneous equation $y'' + 3y' + 2 = 0$ has characteristic equation

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0,$$

with zeros $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, the homogeneous solution is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

For $y_p = Ae^{-4t}$, $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$. Substituting into the inhomogeneous ODE, we get

$$16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t} \implies 6A = 3 \implies A = \frac{1}{2}$$

Thus, a particular solution is $y_p(t) = \frac{1}{2}e^{-4t}$. By Theorem 5.2, the general solution is

$$\boxed{y = C_1 e^{-t} + C_2 e^{-2t} + \tfrac{1}{2}e^{-4t}}$$

The given initial conditions imply:

$$y(0) = C_1 + C_2 + \frac{1}{2} = 1, \quad y'(0) = -C_1 - 2C_2 - 2 = 0 \qquad \implies \quad C_1 = 3, \; C_2 = -5/2.$$

So, the solution to the initial value problem is $\boxed{y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}}$

4.5:26; 10pts: *In the equation* $y'' + 4y = 4\cos 2t$*, the forcing term is also a solution of the associated homogeneous equation. Use this to find a particular solution.*
Our strategy is to look at the equation $z'' + 4z = e^{2it}$, of which the given equation is the real part. The characteristic equation of the homogeneous equation $z'' + 4z = 0$ is $\lambda^2 + 4 = 0$. Its roots are $\pm 2i$. So, the homogeneous solution is:

$$z_h = C_1 e^{2it} + C_2 e^{-2it}$$

The forcing term of $z'' + 4z = 4e^{2it}$ is also a solution of the homogeneous equation. Thus, we try to find a particular solution of the form $z_p = Ate^{2it}$:

$$z_p = Ate^{2it} \implies z_p' = Ae^{2it}(1 + 2it) \implies z_p'' = 4Ae^{2it}(i - t).$$

After substituting these into $z'' + 4z = 4e^{2it}$, we get:

$$4Ae^{2it}(i - t) + 4Ate^{2it} = 4e^{2it} \implies 4iA = 4 \implies A = \frac{1}{i} = -i$$

$$\implies z_p = -ite^{2it} = -it(\cos 2t + i \sin 2t) = t \sin 2t - it \cos 2t.$$

Its real part is a particular solution we are looking for: $\boxed{y_p = Re(z_p) = t\sin 2t}$

4.5:30; 10pts: *If $y_f(t)$ and $y_g(t)$ are solutions of*

$$y'' + py' + qy = f(t) \qquad \textit{and} \qquad y'' + py' + qy = g(t),$$

respectively, show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of

$$y'' + py' + qy = \alpha f(t) + \beta g(t),$$

where α and β are any real numbers.
We are given that:

$$y_f'' + py_f' + qy_f = f(t) \qquad \text{and} \qquad y_g'' + py_g' + qy_g = g(t)$$

We plug in $z(t)$ into $y''+py'+qy=\alpha f(t)+\beta g(t)$ and use these two properties of y_f and y_g:

$$\begin{aligned} z'' + pz' + qz &= (\alpha y_f + \beta y_g)'' + p(\alpha y_f + \beta y_g)' + q(\alpha y_f + \beta y_g) \\ &= (\alpha y_f'' + \beta y_g'') + p(\alpha y_f' + \beta y_g') + q(\alpha y_f + \beta y_g) \\ &= \alpha(y_f'' + py_f' + qy_f) + \beta(y_g'' + py_g' + qy_g) \\ &= \alpha f(t) + \beta g(t). \end{aligned}$$

Thus, $z(t)=\alpha y_f(t)+\beta y_g(t)$ is a solution of $y''+py'+qy=\alpha f(t)+\beta g(t)$.

4.5:32; 12pts: *Use the previous exercise to find a particular solution of the equation*

$$y'' - y = t - e^{-t}.$$

The forcing term is the linear combination $t - e^{-t} = 1 \cdot t + (-1)e^{-t}$. We first find a particular solution y_{p_1} of $y'' - y = t$, and then a particular solution y_{p_2} of $y'' - y = -e^{-t}$. By the previous exercise, $y_{p_1} - y_{p_2}$ will be a particular solution to our equation. To find y_{p_1}, substitute $y = at+b$ into

$$y'' - y = t \implies -at - b = t \implies a = -1,\ b = 0, \implies y_{p_1}(t) = -t.$$

To find y_{p_2}, note that the characteristic equation for the homogeneous equation $y''-y=0$ is $\lambda^2-1=0$. Its roots are $\lambda_1 = -1$ and $\lambda_2 = 1$, giving the homogeneous solution

$$y_h = C_1 e^{-t} + C_2 e^t.$$

It follows that the forcing term e^{-t} is a solution of the homogeneous equation. So we try to find y_{p_2} of the form $y_{p_2}(t) = Ate^{-t}$:

$$y_{p_2} = Ate^{-t} \implies y_{p_2}' = Ae^{-t}(1-t) \implies y_{p_2}'' = Ae^{-t}(t-2).$$

The equation now becomes:

$$e^{-t} = y_{p_2}'' - y_{p_2} = Ae^{-t}(t-2) - Ate^{-t} \implies -2A = 1 \implies A = -\frac{1}{2} \implies y_{p_2}(t) = -\frac{1}{2}te^{-t}.$$

So a particular solution of $y''-y=t-e^{-t}$ is $\boxed{y_p = y_{p1}-y_{p2} = -t+\frac{1}{2}te^{-t}}$

4.5:42; 12pts: *Find a particular solution of the equation* $y''+5y'+4y=te^{-t}$.
The characteristic equation for the corresponding homogeneous equation $y''+5y'+4=0$ is

$$\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4)=0.$$

Its are roots $\lambda_1=-1$ and $\lambda_2=-4$, and the homogeneous solution is

$$y_h=C_1e^{-4t}+C_2e^{-t}.$$

In particular, e^{-t} is a solution to the homogeneous equation. Thus, we modify the hint in Exercise 39, and look for a particular solution of the form $y_p=t(at+b)e^{-t}$:

$$\begin{aligned} y_p(t)=t(at+b)e^{-t} \quad &\Longrightarrow \quad y_p'(t)=(-at^2+(2a-b)t+b)e^{-t} \\ &\Longrightarrow \quad y_p''(t)=(at^2+(-4a+b)t+(2a-2b))e^{-t} \end{aligned}$$

Substituting, we get:

$$te^{-t}=y''+5y'+4y=(6at+(2a+3b))e^{-t} \quad\Longrightarrow\quad 6a=1,\ 2a+3b=0, \quad\Longrightarrow\quad a=\frac{1}{6},\ b=-\frac{1}{9}.$$

Thus, a solution of $y''+5y'+4y=te^{-t}$ is $\boxed{y_p=\frac{1}{6}t^2e^{-t}-\frac{1}{9}te^{-t}}$

Section 4.6, Problem 13

Verify that $y_1(t)=t$ *and* $y_2(t)=t^{-3}$ *are solutions to the homogeneous equation*

$$t^2y''+3ty'-3y=0.$$

Use variation of parameters to find the general solution to

$$t^2y''+3ty'-3y=t^{-1}.$$

For the first part, plug in $y_1(t)=t$ and $y_2(t)=t^{-3}$ into the homogeneous equation:

$$y_1=t,\ y_1'=1,\ y_1''=0 \quad\Longrightarrow\quad t^2y_1''+3ty_1'-3y_1=t^2\cdot 0+3t\cdot 1-3\cdot t=0;$$
$$y_1=t^{-3},\ y_1'=-3t^{-4},\ y_1''=12t^{-5} \quad\Longrightarrow\quad t^2y_2''+3ty_2'-3y_2=t^2\cdot(12t^{-5})+3t\cdot(-3t^{-4})-3t^{-3}=0,$$

as needed. We look for a solution to the inhomogeneous equation of the form $y_p=v_1y_1+v_2y_2$. Then,

$$y_p'=(y_1v_1'+y_2v_2')+y_1'v_1+y_2'v_2=(tv_1'+t^{-3}v_2')+v_1-3t^{-4}v_2.$$

We set the expression in the parenthesis to zero. Thus,

$$y_p'=v_1-3t^{-4}v_2 \quad\Longrightarrow\quad y_p''=v_1'+12t^{-5}v_2-3t^{-4}v_2' \quad\Longrightarrow\quad t^2y_p''+3ty_p'-3y_p=t^2v_1'-3t^{-2}v_2'=t^{-1}.$$

Since we also assumed that $tv_1'+t^{-3}v_2'=0$, we need to solve the system

$$\begin{cases} v_1'+t^{-4}v_2'=0 \\ v_1'-3t^{-4}v_2'=t^{-3} \end{cases} \quad\Longrightarrow\quad v_1'=\frac{1}{4}t^{-3},\ v_2'=-\frac{1}{4}t \quad\Longrightarrow\quad v_1=-\frac{1}{8}t^{-2},\ v_2=-\frac{1}{8}t^2$$
$$\Longrightarrow\quad y_p=v_1y_1+v_2y_2=-\frac{1}{8}t^{-2}\cdot t-\frac{1}{8}t^2\cdot t^{-3}=-\frac{1}{4}t^{-1}.$$

Thus, the general solution is $\boxed{y(t)=C_1t+C_2t^{-3}-\frac{1}{4}t^{-1}}$