MAT531 Geometry/Topology Homework 7

1. Using the Mayer-Vietoris exact sequence, find the Betti numbers (the dimensions of the cohomology spaces) for the punctured torus (= a torus with a hole, the boundary of the hole being removed).

2. Find the Betti numbers of the sphere with two handles.

3. Find the Betti numbers of the sphere with $g$ handles (the number $g$ is called the genus of this surface).

4. Consider an exact sequence of vector spaces:

$$0 \to V^0 \to V^1 \to V^2 \to V^3 \to \cdots$$

Prove that $\sum_{k=0}^{\infty} (-1)^k \dim(V^k) = 0$ provided that this sum is finite.

5. Let $V$ be a complex of finite dimensional vector spaces. The Euler characteristic of $V$ is defined as

$$\chi(V) = \sum_{k=0}^{\infty} (-1)^k \dim(H^k(V))$$

provided that the sum in the right hand side is finite. Consider a short exact sequence of complexes

$$0 \to U^0 \to V^0 \to W^0 \to 0$$

Show that $\chi(U^0) - \chi(V^0) + \chi(W^0) = 0$.

6. The Euler characteristic of a manifold is defined as the Euler characteristic of its de Rham co-chain complex (i.e. as the alternating sum of its Betti numbers). Prove the additivity of the Euler characteristic: if $U$ and $V$ are open subsets of a manifold $X$ such that $X = U \cap V$, then

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$