MAT531 Geometry/Topology Homework 7

1. Using the Mayer-Vietoris exact sequence, find the *Betti numbers* (the dimensions of the cohomology spaces) for the punctured torus (= a torus with a hole, the boundary of the hole being removed).

2. Find the Betti numbers of the sphere with two handles.

3. Find the Betti numbers of the sphere with g handles (the number g is called the *genus* of this surface).

4. Consider an exact sequence of vector spaces:

$$0 \to V^0 \to V^1 \to V^2 \to V^3 \to \cdots$$

Prove that $\sum_{k=0}^{\infty}(-1)^k \dim(V^k) = 0$ provided that this sum is finite.

5. Let $V^\cdot$ be a complex of finite dimensional vector spaces. The *Euler characteristic* of $V^\cdot$ is defined as

$$\chi(V^\cdot) = \sum_{k=0}^{\infty}(-1)^k \dim(H^k(V^\cdot))$$

provided that the sum in the right hand side is finite. Consider a short exact sequence of complexes

$$0 \to U^\cdot \to V^\cdot \to W^\cdot \to 0$$

Show that $\chi(U^\cdot) - \chi(V^\cdot) + \chi(W^\cdot) = 0$.

6. The *Euler characteristic of a manifold* is defined as the Euler characteristic of its de Rham co-chain complex (i.e. as the alternating sum of its Betti numbers). Prove the *additivity of the Euler characteristic*: if U and V are open subsets of a manifold X such that $X = U \cap V$, then

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$