MAT 531 Geometry/Topology Homework 4

1. Using the coordinate expression for the differential of a $k$-form
\[ \alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \implies d\alpha = d\alpha_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \]
prove that $d(d\alpha) = 0$ and that
\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta. \]

Recall the coordinate expressions for the gradient of a function $f$ on $\mathbb{R}^3$, and the divergence and the curl of a vector field $v$ on $\mathbb{R}^3$:
\[ \text{grad}(f) = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix}, \quad \text{div}(v) = \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3, \quad \text{curl}(v) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v^1 & v^2 & v^3 \end{pmatrix}. \]
Here $(e_1, e_2, e_3)$ is an orthonormal basis for $\mathbb{R}^3$, $\partial_1, \partial_2, \partial_3$ the corresponding differentiations (partial derivatives), and $(v^1, v^2, v^3)$ coordinates of $v$ in the given basis.

2. Prove that for any vector field $u$ on the space $\mathbb{R}^3$ with a fixed Euclidean metric and for any function $f : \mathbb{R}^3 \to \mathbb{R}$,
\[ \langle \text{grad}(f), u \rangle = d\alpha(u) = \partial_u f. \]
(the last equality is independent of a Euclidean metric).

3. In the settings of Problem 2, prove that $\text{div}(u) = d\alpha(e_1, e_2, e_3)$, where the differential 2-form $\alpha$ is the unique 2-form on $\mathbb{R}^3$ satisfying the relation
\[ \alpha(v, w) = \det(u, v, w) = \det \begin{pmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{pmatrix} \]
for any vector fields $v$ and $w$ on $\mathbb{R}^3$.

4. In the same settings, prove that $\det(\text{curl}(u), v, w) = d\beta(v, w)$, where the differential 1-form $\beta$ is the unique 1-form on $\mathbb{R}^3$ satisfying the relation $\langle u, v \rangle = \beta(v)$ for any vector field $v$ on $\mathbb{R}^3$.

5. Prove the following relations:
\[ \text{curl}(\text{grad}(f)) = 0, \quad \text{div}(\text{curl}(v)) = 0, \quad \text{div}(\text{grad}(f)) = \Delta f, \]
\[ \text{curl}(\text{curl}(v)) = \text{grad}(\text{div}(v)) - \Delta v. \]
The Laplace operator $\Delta$ is defined as $\partial_1^2 + \partial_2^2 + \partial_3^2$.

6*. For any 1-form $\alpha$ on a smooth manifold, and for any pair of vector fields $v$ and $w$, prove that
\[ d\alpha(v, w) = \partial_v \alpha(w) - \partial_w \alpha(v) - \alpha([v, w]). \]