MAT 531 Geometry/Topology Homework 4

1. Using the coordinate expression for the differential of a k-form

$$\alpha = \alpha_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \qquad \Longrightarrow \qquad d\alpha = d\alpha_{i_1\dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

prove that $d(d\alpha) = 0$ and that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta.$$

Recall the coordinate expressions for the gradient of a function f on $\mathbb{R}^3$, and the divergence and the curl of a vector field v on $\mathbb{R}^3$:

$$\operatorname{grad}(f) = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix}, \quad \operatorname{div}(v) = \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3, \quad \operatorname{curl}(v) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v^1 & v^2 & v^3 \end{pmatrix}.$$

Here (e_1, e_2, e_3) is an orthonormal basis for $\mathbb{R}^3$, $\partial_1, \partial_2, \partial_3$ the corresponding differentiations (partial derivatives), and (v^1, v^2, v^3) coordinates of v in the given basis.

2. Prove that for any vector field u on the space $\mathbb{R}^3$ with a fixed Euclidean metric and for any function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\langle \operatorname{grad} f, u \rangle = df(u) = \partial_u f.$$

(the last equality is independent of a Euclidean metric).

3. In the settings of Problem 2, prove that $\operatorname{div}(u) = d\alpha(e_1, e_2, e_3)$, where the differential 2-form α is the unique 2-form on $\mathbb{R}^3$ satisfying the relation

$$\alpha(v, w) = \det(u, v, w) = \det \begin{pmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{pmatrix}$$

for any vector fields v and w on $\mathbb{R}^3$.

4. In the same settings, prove that $\det(\operatorname{curl}(u), v, w) = d\beta(v, w)$, where the differential 1-form β is the unique 1-form on $\mathbb{R}^3$ satisfying the relation $\langle u, v \rangle = \beta(v)$ for any vector field v on $\mathbb{R}^3$.

5. Prove the following relations:

$$\operatorname{curl}(\operatorname{grad} f) = 0, \quad \operatorname{div}(\operatorname{curl}(v)) = 0, \quad \operatorname{div}(\operatorname{grad} f) = \Delta f,$$
$$\operatorname{curl}(\operatorname{curl}(v)) = \operatorname{grad}(\operatorname{div}(v)) - \Delta v.$$

The Laplace operator Δ is defined as $\partial_1^2 + \partial_2^2 + \partial_3^2$.

6*. For any 1-form α on a smooth manifold, and for any pair of vector fields v and w, prove that

$$d\alpha(v, w) = \partial_v \alpha(w) - \partial_w \alpha(v) - \alpha([v, w]).$$