MAT 530: Topology & Geometry, I  
Fall 2005

Problem Set 10

Solution to Problem p483, #2

(a) Show that every continuous map \( f: \mathbb{R}P^2 \rightarrow S^1 \) is null-homotopic.

(b) Find a continuous map from the 2-dimensional torus to \( S^1 \) which is not null-homotopic.

(a) Let \( q: \mathbb{R} \rightarrow S^1 \), \( q(s) = e^{2\pi is} \), be the standard covering map. Fix \( x_0 \in \mathbb{R}P^2 \). Let \( b_0 = f(x_0) \). Choose \( e_0 \in q^{-1}(b_0) \). Since

\[
\pi_1(\mathbb{R}P^2, x_0) = \mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_1(S^1, b_0) = \mathbb{Z},
\]

the homomorphism \( f_*: \pi_1(\mathbb{R}P^2, x_0) \rightarrow \pi_1(S^1, b_0) \) must be trivial. Thus,

\[
f_*\pi_1(\mathbb{R}P^2, x_0) \subset q_*\pi_1(\mathbb{R}, e_0) \subset \pi_1(S^1, b_0)
\]

and by the General Lifting Lemma the map \( f: (\mathbb{R}P^2, x_0) \rightarrow (S^1, b_0) \) lifts to a continuous map

\[
\tilde{f}: (\mathbb{R}P^2, x_0) \rightarrow (\mathbb{R}, e_0),
\]

i.e. \( f = \tilde{f} \circ q \) as indicated in Figure 1. Since \( \mathbb{R} \) is contractible, \( \tilde{f} \) is null-homotopic. If \( \tilde{H} \) is a homotopy from \( \tilde{f} \) to the map sending \( \mathbb{R}P^2 \) to some \( e \in \mathbb{R} \), then \( \tilde{H} \equiv q \circ \tilde{H} \) is a homotopy from \( f \) to the map sending \( \mathbb{R}P^2 \) to \( q(e) \in S^1 \). Thus, \( f \) is null-homotopic.

Figure 1: Diagrams for p483, #2

Remark: This argument depends only on the facts that the homomorphism \( f_* \) between the fundamental groups of the domain and the target of \( f \) is trivial and that the universal cover of the target is contractible.

(b) The 2-dimensional torus \( T \) is homeomorphic to \( S^1 \times S^1 \). Let

\[
\pi_1: (T, x_0 \times y_0) \rightarrow (S^1, x_0)
\]

be the projection onto the first component. Since the homomorphism

\[
\pi_{1*}: \pi_1(T, x_0 \times y_0) \rightarrow \pi_1(S^1, x_0) \approx \mathbb{Z}
\]

is surjective, it is not trivial. Thus, \( \pi_1 \) is not null-homotopic.
Solution to Problem p483, #5

Suppose \( T = S^1 \times S^1, \ x_0 \in S^1, \) and \( x_0 = b_0 \times b_0. \)

(a) Show that every isomorphism of \( \pi_1(T, x_0) \) with itself is induced by a homeomorphism of \( (T, x_0) \) with itself.

(b) If \( E \) is a covering space of \( T, \) then \( E \) is homeomorphic to \( \mathbb{R}^2, \ S^1 \times \mathbb{R}, \) or \( S^1 \times S^1. \)

(a) Let

\[
q: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad q(u, v) = (e^{2\pi i u}, e^{2\pi i v}),
\]

be the standard covering map. We assume that \( x_0 = q(0). \) Let

\[
\tilde{\alpha}, \tilde{\beta}: I \longrightarrow \mathbb{R}^2, \quad \tilde{\alpha}(s) = (s, 0) \quad \text{and} \quad \tilde{\beta}(s) = (0, s),
\]

be the horizontal path running from 0 to (1, 0) and the vertical path running from 0 to (0, 1). The group \( \pi_1(T, x_0) \) is the free abelian group generated by the loops

\[
\alpha \equiv q \circ \tilde{\alpha} \quad \text{and} \quad \beta \equiv q \circ \tilde{\beta}, \quad \text{i.e.} \quad \pi_1(T, x_0) = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta].
\]

Under this isomorphism, the class \( m \alpha + n \beta \) corresponds to the equivalence class of the loop \( \gamma \equiv q \circ \tilde{\gamma} \) in \( T, \) where \( \tilde{\gamma} \) is any path in \( X \) from 0 to \( (m, n) \in \mathbb{Z}^2 \subset \mathbb{R}^2. \) Via this identification, an isomorphism \( T \) of \( \pi_1(T, x_0) \) with itself corresponds to a \( 2 \times 2 \) integer matrix

\[
A: \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta] \longrightarrow \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]
\]

which has an integer matrix inverse \( B. \) Define

\[
\tilde{h}_A, \tilde{h}_B: (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0) \quad \text{by} \quad \tilde{h}_A(v) = Av \quad \text{and} \quad \tilde{h}_B(v) = Bv.
\]

Since the maps

\[
q \circ \tilde{h}_A, q \circ \tilde{h}_B: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow T
\]

are constant along the fibers of quotient projection map \( q: \mathbb{R}^2 \longrightarrow T, \) they induce maps

\[
h_A, h_B: T \longrightarrow T \quad \text{s.t} \quad q \circ \tilde{h}_A = h_A \circ q \quad \text{and} \quad q \circ \tilde{h}_B = h_B \circ q;
\]

see Figure 2.

Figure 2: Diagrams for p483, #5a

Since \( q \) (on the left of each diagram) is a quotient map and the maps \( q \circ \tilde{h}_A \) and \( q \circ \tilde{h}_B \) are continuous (and open), so are the maps \( h_A \) and \( h_B. \) Since

\[
\tilde{h}_A \circ \tilde{h}_B = AB = \text{Id} = \text{id}_{\mathbb{R}^2} \quad \text{and} \quad \tilde{h}_B \circ \tilde{h}_A = BA = \text{Id} = \text{id}_{\mathbb{R}^2},
\]

Since

\[
\tilde{h}_A \circ \tilde{h}_B = AB = \text{Id} = \text{id}_{\mathbb{R}^2} \quad \text{and} \quad \tilde{h}_B \circ \tilde{h}_A = BA = \text{Id} = \text{id}_{\mathbb{R}^2},
\]
\[ h_A \circ h_B = \text{id}_T \] and \[ h_B \circ h_A = \text{id}_T. \] In particular, \( h_A : (T, x_0) \to (T, x_0) \) is a homeomorphism. Since
\[
h_A(0) = 0 \quad q \circ h_A \circ \alpha = h_A \circ q \circ \alpha, \quad q \circ h_A \circ \beta = h_A \circ q \circ \beta,
\]
\[
\{h_A \circ \alpha\}(1) = A\alpha(1) = A(1, 0)^t, \quad \{h_A \circ \beta\}(1) = A\beta(1) = A(0, 1)^t,
\]
it follows that
\[
h_{A^*}[\alpha] = [q \circ h_A \circ \alpha] = A(1, 0)^t = T\alpha \quad \text{and} \quad h_{A^*}[\beta] = [q \circ h_A \circ \beta] = A(0, 1)^t = T\beta.
\]
Thus, \( h_* = T \) as needed.

(b) If \( p : E \to T \) is covering map, choose \( e_0 \in p^{-1}(x_0) \). Let
\[
H_{e_0} = p_*\pi_1(E, e_0) \subset \pi_1(T, x_0).
\]
Since \( \pi_1(T, x_0) \) is a free abelian group of rank 2, there exists a basis \( \{ e_1, e_2 \} \) for \( \pi_1(T, x_0) \) such that
\begin{itemize}
  \item[(i)] \( \{ me_1, ne_2 \} \) is a basis for \( H_{e_0} \) for some \( m, n \in \mathbb{Z}^+ \), or
  \item[(ii)] \( \{ me_1 \} \) is a basis for \( H_{e_0} \) for some \( m \in \mathbb{Z}^+ \), or
  \item[(iii)] \( H_{e_0} = \{ 0 \} \).
\end{itemize}
By part (a), there exists a homeomorphism \( h : (T, x_0) \to (T, x_0) \) such that
\[
h_*e_1 = [\alpha] \quad \text{and} \quad h_*e_2 = [\beta].
\]
Then, \( p' = h \circ p : (E, e_0) \to (T, x_0) \) is a covering map and
\begin{itemize}
  \item[(i)] \( \{ m\alpha, n\beta \} \) is a basis for \( p'_*\pi_1(E, e_0) \) for some \( m, n \in \mathbb{Z}^+ \), or
  \item[(ii)] \( \{ m\alpha \} \) is a basis for \( p'_*\pi_1(E, e_0) \) for some \( m \in \mathbb{Z}^+ \), or
  \item[(iii)] \( p'_*\pi_1(E, e_0) = \{ 0 \} \).
\end{itemize}
Let \( \tilde{p} : (\tilde{E}, \tilde{e}_0) \to (T, x_0) \), be the covering map given by
\begin{itemize}
  \item[(i)] \( \tilde{E} = S^1 \times S^1, \tilde{p}(w, z) = (w^m, z^n) \implies \tilde{p}_*\pi_1(\tilde{E}, \tilde{e}_0) = \mathbb{Z}\{ m\alpha, n\beta \}; \)
  \item[(ii)] \( \tilde{E} = S^1 \times \mathbb{R}, \tilde{p}(w, v) = (w^m, e^{2\pi i v}) \implies \tilde{p}_*\pi_1(\tilde{E}, \tilde{e}_0) = \mathbb{Z}\{ m\alpha \}; \)
  \item[(iii)] \( \tilde{E} = \mathbb{R} \times \mathbb{R}, \tilde{p}(u, v) = (e^{2\pi i u}, e^{2\pi i v}) \implies \tilde{p}_*\pi_1(\tilde{E}, \tilde{e}_0) = \{ 0 \}. \)
\end{itemize}
Since \( p'_*\pi_1(E, e_0) = p_*\pi_1(\tilde{E}, \tilde{e}_0) \subset \pi_1(T, x_0) \), the covering maps \( (E, p', T) \) and \( (\tilde{E}, \tilde{p}, T) \) are equivalent. In particular, there exists a homeomorphism \( g : E \to \tilde{E} \). Thus, \( E \) is homeomorphic to \( S^1 \times S^1 \), \( S^1 \times \mathbb{R} \), or \( \mathbb{R}^2 \) depending on the case.

**Solution to Problem p493, #5**

If \( n \) and \( k \) are relatively prime positive numbers, let \( h \) be the map of \( S^3 \subset \mathbb{C}^2 \) to itself given by
\[
h : S^3 \to S^3, \quad h(z_1, z_2) = (e^{2\pi i/n}z_1, e^{2\pi i k/n}z_2).
\]

(a) Show that \( h \) generates a subgroup \( G \) of the homeomorphism group of \( S^3 \) and that the action of \( G \) is fixed-point free.

(b) The lens space \( L(n, k) \) is the quotient space \( S^3/G \). Show that if \( L(n, k) \) is homeomorphic to \( L(n', k') \), then \( n = n' \).

(c) Show that \( L(n, k) \) is a (smooth) compact 3-manifold.
(a) If \( m \in \mathbb{Z}^+ \), then the map \( h^m \) is given by

\[
h^m: S^3 \rightarrow S^3, \quad h(z_1, z_2) = (e^{2\pi im/n} z_1, e^{2\pi ikm/n} z_2).
\]

In particular, \( h^n \) is the identity map on \( S^3 \) and thus \((h^m)^{-1} = h^{n-m}\). Since \( h \) is continuous, so are its composites \( h^m \). Thus, all maps \( h^m \) are homeomorphisms, and \( h \) generates a subgroup \( G \) of the homeomorphism group of \( S^3 \). Furthermore,

\[
h^m(z_1, z_2) = (z_1, z_2) \quad \forall \ (z_1, z_2) \in S^3 \quad \iff \quad e^{2\pi im/n} z_1 = z_1, \ e^{2\pi ikm/n} z_2 = z_2 \quad \forall \ (z_1, z_2) \in S^3
\]

Thus, \( h^m = \text{id} \) if and only if \( m \) is divisible by \( n \), i.e. \( G \) is a cyclic group of order \( n \). Finally, if \((z_1, z_2) \in S^3\),

\[
h^m(z_1, z_2) = (z_1, z_2) \quad \iff \quad e^{2\pi im/n} z_1 = z_1, \ e^{2\pi ikm/n} z_2 = z_2.
\]

Since either \( z_1 \neq 0 \) or \( z_2 \neq 0 \), this implies that either \( e^{2\pi im/n} = 1 \) or \( e^{2\pi ikm/n} = 1 \). Since \( k \) and \( n \) are relatively prime, the two conditions are equivalent to \( m \) being divisible by \( n \), i.e. \( h^m = \text{id} \). So, the action of \( G \) is fixed point-free.

(b) Since the action of the finite group \( G \) on the Hausdorff space \( S^3 \) is fixed-point free, by p493, #4 this action is properly discontinuous as well. Thus, the quotient map

\[
q: S^3 \rightarrow L(n, k) = S^3/G
\]

is a covering map. Since \( S^3 \) is simply connected,

\[
\pi_1(L(n, k), x_0) = G \approx \mathbb{Z}_n \equiv \mathbb{Z}/n\mathbb{Z}.
\]

If \( L(n, k) \) is homeomorphic to \( L(n', k') \), \( \pi_1(L(n, k), x_0) \) and \( \pi_1(L(n', k'), x'_0) \) must be isomorphic and in particular must have the same cardinality, i.e. \( n = n' \).

(c) Since the map \( q \) in part (b) is a quotient map and \( S^3 \) is compact, so is \( L(n, k) \). Since \( q \) is a covering map and \( S^3 \) is Hausdorff, so is \( L(n, k) \). If \( V \subset L(n, k) \) is an open set evenly covered by \( q \), \( V \) is homeomorphic to a proper open subset \( U \) of \( S^3 \). Since \( S^3 - \{pt\} \) is homeomorphic to \( \mathbb{R}^3 \), it follows that every point in \( L(n, k) \) has a neighborhood homeomorphic to an open subset of \( \mathbb{R}^3 \). We conclude that \( S^3 \) is a compact topological 3-manifold.

Remark: In contrast to higher-dimensional manifolds, every 3-dimensional topological manifold admits a unique smooth structure (this was proved by Munkres). So, \( L(n, k) \) with its unique smooth structure is a compact smooth 3-manifold. In fact, \( h \) is a diffeomorphism of \( S^3 \). Thus, the group \( G \) is a subgroup of the diffeomorphism group of \( S^3 \) and the smooth structure on \( S^3 \) induces a smooth structure on \( L(n, k) \).