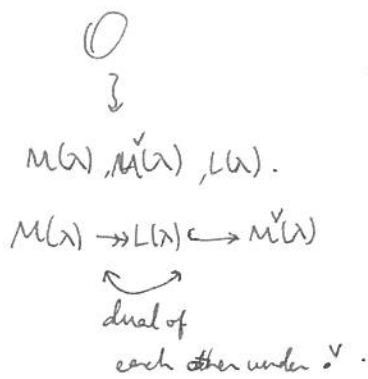


Lie 7/12/17

Recall the story so far...



Algebraic part.

- Cartan acts ss.

- $Z(\mathfrak{U}\mathfrak{g})$ may not

$\mathcal{O} \rightarrow$ lift can always lift actions of \mathfrak{g} to B

$\tilde{\mathcal{O}}_\lambda \rightarrow \dots \mathfrak{g}$ to N only \dots

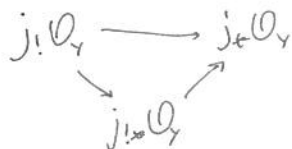
But... $\mathcal{O} \sim \tilde{\mathcal{O}}$ eq! |

D-modules $\mathcal{G}_B = X$

Geometric part

$Y \hookrightarrow X$ sm. emb. subvar. (locally closed).

$j_*\mathcal{O}_Y, j_!\mathcal{O}_Y \xrightarrow{\text{(hol.)}} \mathcal{D}\text{-mod. supp on } Y.$



Then: $j_{!*}\mathcal{O}_Y$ is simple (as obj. in $\text{cat Mod}_h(X)$).

Eg. $X = \mathbb{C}, Y = \mathbb{C} - \{0\}$.

$j_*\mathbb{C}[x, x^{-1}] = \mathbb{C}[x, x^{-1}]$

$j_!\mathbb{C}[x, x^{-1}] = \mathbb{D}_x \cdot \int \delta$

$j_{!*}\mathbb{C}[x, x^{-1}] = \mathbb{C}[x]$

• $\mathcal{D}_*(\mathcal{G}_B) = \mathfrak{U}\mathfrak{g} / \ker \chi.$

• \mathcal{G}_B is D-affine $(\Gamma: \text{Mod}_{\mathfrak{g}, \mathbb{C}}(\mathcal{D}\mathcal{D}_{\mathcal{G}_B}) \rightarrow \text{Mod}(\frac{\mathcal{D}(\mathcal{G}_B)}{\mathbb{C}}))$ is an equiv. of cat.

$\tilde{\mathcal{O}}_\lambda$
 $Z(\mathfrak{U}\mathfrak{g})$ acts ss.
 Cartan doesn't have

Cor: $\text{Mod}(\mathbb{A} \text{ } \mathcal{D}_{G/B}) \cong \text{Rep}(g, \chi_0) = \{V \in \text{Rep}(g) : \exists |v = \chi_0(z) \text{ id}\}$.

This brings us to algebra land. Note difference: $\text{Rep}(g, \chi_0)$ — no restr. on h.w.

$\text{Rep}(g)$ in \mathcal{O} — no restr. on χ_0 ,
 & how $\mathbb{Z}(Ug)$ ac
 (may not act by scalar)

Let $\mathcal{C} = \text{set. of } N_+ \text{ - eq. } \mathcal{D}_{G/B} \text{ - mod. w/ Reg. sing.}$
 holon.

$N \setminus G/B \cong \mathbb{A}^1 \setminus G/B$ as sets.
 \cong
 = \mathcal{D} -mod. "constant" along Bruhat cells $X_w = \text{orbit of } w \in B \text{ in } G/B$.
 = \mathcal{D} -mod. which have comp. series w/ all factors of the form
 $L_w = j_{!*}(\mathcal{O}_{X_w})$.
 "const."

Capital T Theorem

$$\mathcal{C} \cong \widetilde{\mathcal{O}}_0$$

moreover,

$$j_* \mathcal{O}_{X_w} \longrightarrow M^v(w_0 w. 0)$$

$$j_! \mathcal{O}_{X_w} \longrightarrow M(w_0 w. 0)$$

$$j_{!*} \mathcal{O}_{X_w} \longrightarrow L(w_0 w. 0)$$

when $\overline{Y_w}$ is smooth, life is easy:

eg. $j_{!*} \mathcal{O}_{X_0} = \mathcal{O}_X$ if
 "open cell"

$$\Gamma \cong \mathcal{C} = L(0).$$

Solving the geometric problem of studying $j_! j^* \mathcal{F}$ on G/B .

Analytic situation: vector bundles V w/ flat connection ∇ .

flat sections of $V = \{s \in V \mid \nabla s = 0\}$.

$$\nabla \text{ flat} \iff \text{ex } 0 \rightarrow V \xrightarrow{d} V \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{d} V \otimes_{\mathcal{O}_X} \Omega_X^2 \rightarrow \dots$$

(sheaf of local sections)

$$H^i \text{ flat sections} = H^i(V \otimes_{\mathcal{O}_X} \Omega_X^i)$$

$$DR(V) = (V \otimes_{\mathcal{O}_X} \Omega_X^i)$$

Then $H^i(DR(V))$ is locally constant.

Can repeat this for any D -mod. and gives us a functor

$$DR: \underset{\text{hol.}}{\mathcal{D}_X} \text{Mod}(\mathcal{D}_X) \rightarrow \underset{\text{on } X}{\mathcal{D}}(\text{Constructible sheaves})$$

ie. ex of sheaves st-

$H^i(M^0)$ are const.

→ there is stratification of X

Then (Riemann-Hilbert correspondence).

$$X = \coprod X_i, \text{ st.}$$

restr. of \mathcal{F} to X_i is locally const.

DR is equivalence

$$D_{RS}^b(\mathcal{D}_X) \rightarrow D_{\text{const.}}^b(X)$$

where RS is "Reg. sing."

(this is a far-reaching generalization of (in analytic situation) V flat conn \longleftrightarrow locally const sheaves.)

note: this doesn't work for alg. situation:

there is V on $\mathbb{C} \setminus 0$ that can't be written as locally const sheaf w/ trivial monodromy

but bundle w/ flat connection that can't be under algebraic change of coordinate.

Repeating the same question (of multiplicities) in $D_{\text{const}}^b(X_B)$

gives Kashiwara-Lauderbach theorem:

There is a family of polynomials $P_{w, w'}(q)$ uniquely determined by

conditions • $P_{w, w'} = 0$ unless $w' \leq w$. $P_{w, w} = 1$

• $C_w = \sum_{w' \leq w} q^{\dots} P_{w, w'}(q) T_{w'} \in \text{Hecke algebra}$.

satisfies $\overline{C}_w = C_w$

↓ deformation of $\mathbb{C}[W]$.
by formal q .
↓
algebra over $\mathbb{C}[q, q^{-1}]$. T_w

Then: $P_{w, w'}(q=1)$ gives multiplicities

$$[L(w, \lambda) : M(w', \lambda)]$$

$\mathbb{F} \cong \mathbb{Z}$

$\cdot : H \rightarrow H$

$q \mapsto q^{-1}$

(in $D_{\text{const}}^b(X)$, there is finer str.)

- consider this setup in $\overline{\mathbb{F}_p}$,

of grading on stalks given by Frobenius action on $\overline{\mathbb{F}_p}$

- q keeps track of this grading ...]