Lie 7/12/17 (Dec)

Recall the story so far ...

$\mathcal{O}$
$\wr$

$\lambda \in P_+ \rightsquigarrow \mathcal{O}_\lambda$.

$M(\lambda), M^\vee(\lambda), L(\lambda)$.

compute multiplicities

$M(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow M^\vee(\lambda)$

simple obj.: $L(w \cdot \lambda)$ $w \in W$.

dual of each other under $\cdot^\vee$.

$[M(w_1 \cdot \lambda) : L(w_2 \cdot \lambda)] =: c_{w_1 w_2} = ?$

upper Δ wrt. Bruhat order. ind. of λ.

Algebraic part.

- Center acts s.s.
- $Z(Ug)$ may not.

$\tilde{\mathcal{O}}_\lambda$

$Z(Ug)$ acts s.s.
Cartan doesn't have λ

$\mathcal{O}$ — ~~b/f~~ can always lift actions of $\mathfrak{g}$ to B

$\tilde{\mathcal{O}}_\lambda$ — ... $\mathfrak{g}$ to N only ...

$\boxed{\text{But} \ldots\ \mathcal{O} \sim \tilde{\mathcal{O}} \text{ eq!}}$

Geometric part

D-modules $G/B = X$

$Y \overset{j}{\hookrightarrow} X$ ~~sm. ~~ ~~conic~~ subvar. (locally closed).

$j_* \mathcal{O}_Y,\ j_! \mathcal{O}_Y \longrightarrow$ (hol.) D-mod. supp on Y.

$$\begin{tikzcd} j_!\mathcal{O}_Y \arrow[rr] \arrow[dr] & & j_*\mathcal{O}_Y \\ & j_{!*}\mathcal{O}_Y \arrow[ur] & \end{tikzcd}$$

Thm: $j_{!*}\mathcal{O}_Y$ is simple (as obj. in ~~cat~~ $Mod_h(X)$).

Eg. $X = \mathbb{C}$, $Y = \mathbb{C} - \{0\}$.

$j_* \mathbb{C}[x, x^{-1}] = \mathbb{C}[x, x^{-1}]$

$j_! \mathbb{C}[x, x^{-1}] = D_X \cdot \int \delta$

$j_{!*} \mathbb{C}[x, x^{-1}] = \mathbb{C}[x]$.

- $D_{\chi}(G/B) = Ug/\ker \chi_0$.
- G/B is D-affine $\left(\Gamma : Mod_{q.c}(\mathcal{D}_{G/B}) \longrightarrow Mod(D(G/B)) \right)$ is an equiv. of cat.

Cor: $\mathrm{Mod}(\cancel{\mathcal{D}A}\ \mathcal{D}_{G/B}) \simeq \mathrm{Rep}(\mathfrak{g}, \chi_0) = \{V \in \mathrm{Rep}(\mathfrak{g}) : z|_V = \chi_0(z)\,\mathrm{id}\}$.

This brings us to algebra land. Note difference: $\mathrm{Rep}(\mathfrak{g}, \chi_0)$ — no restr. on h.w.

$\cancel{\mathrm{Rep}(\mathfrak{g})}$ in $\mathcal{O}$ — no restr. on χ_0, & how $Z(U\mathfrak{g})$ ac (may not act by scalar

Let $\mathcal{C}$ = cat. of N_+-eq. holon. $\mathcal{D}_{G/B}$-mod. w/ Reg. sing.

$N\backslash G/B \neq B\backslash G/B$ as sets.

= D-mod. "constant" along Bruhat cells Y_w = orbit of w B in G/B.

= D-mod. which have comp. series w/ all factors of the form

$$\mathcal{L}_w = j^w_{!*}(\mathcal{O}_{Y_w}) .$$

"const."

Capital T

Theorem.

$$\mathcal{C} \simeq \widetilde{\mathcal{O}}_0$$

moreover,

$$j_* \mathcal{O}_{Y_w} \longmapsto M^\vee(w_0 w . 0)$$

$$j_! \mathcal{O}_{Y_w} \longmapsto M(w_0 w . 0)$$

$$j_{!*} \mathcal{O}_{Y_w} \longmapsto L(w_0 w . 0) .$$

when $\overline{Y}_w$ is smooth, life is easy:

eg. $j_{!*}\mathcal{O}_{X_0} = \mathcal{O}_X$

open cell

$\mathcal{O} \leadsto \mathbb{C} = L(0)$.

Solving the geometric problem of studying j^*, $j_!$, j_*, $j_{!*}$ on G/B.

Analytic situation : vector bundles V w/ flat connection ∇.

flat sections of V = $\{ s \mid \nabla_i s = 0 \}$.

∇ flat $\iff$ cx $0 \longrightarrow \mathcal{V} \xrightarrow{d} \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{d} \mathcal{V} \otimes \Omega^2_X \longrightarrow \cdots$

sheaf of local sections.

flat sections = $H^0(\mathcal{V} \otimes \Omega^\bullet_X)$

$DR(\mathcal{V}) \coloneqq (\mathcal{V} \otimes \Omega^\bullet_X)$

Then $H^i(DR(\mathcal{V}))$ is locally constant.

Can repeat this for any D-mod. and gives us a functor

$$DR : \underset{\text{hol.}}{D^b} Mod(\mathcal{D}_X) \longrightarrow D\begin{pmatrix}\text{Constructible sheaves} \\ \text{on } X\end{pmatrix}.$$

i.e. cx of sheaves s.t. $H^i(M^\bullet)$ are constr.

$\rightarrow$ there is stratification of X

$X = \coprod X_i$, s.t.

restr. of $\mathcal{F}$ to X_i is locally const.

Thm (Riemann-Hilbert correspondence).

DR is equivalence

$$D^b_{RS}(\mathcal{D}_X) \longrightarrow D^b_{constr.}(X)$$

where RS = "Reg. sing."

this is a far-reaching generalization of $\left(\begin{array}{c} \text{(in analytic situation).} \\ V \text{ flat conn} \longleftrightarrow \text{locally const sheaves.} \end{array} \right)$

note : this doesn't work in alg. situation :

there's V on $\mathbb{C}\setminus 0$ that can't be written as

locally const sheaf w/ trivial monodromy

vect bundle w/ flat connection can't be under algebraic change of coordinate.

Repeating the same question (of multiplicities) in $D^b_{con}(G/B)$

gives Kazhdan–Lusztig thm :

There is a family of polynomials $P_{w,w'}(q)$ uniquely determined by

conditions • $P_{w,w'} = 0$ unless $w' \leq w$. $P_{ww} = 1$

• $C_w = \sum_{w' \leq w} q^{\cdots} P_{ww'}(q) T_{w'} \in$ Hecke algebra.

satisfies $\overline{C_w} = C_w$

deformation of $\mathbb{C}[W]$. w

by formal q.

algebra over $\mathbb{C}[q, q^{-1}]$. T_w

$\overline{\cdot} : H \to H$

$q \mapsto q^{-1}$.

Thm : $P_{w,w'}(q=1)$ gives multiplicities

$[L(w.\lambda) : M(w'.\lambda)]$

[in $D^b_{const.}(X)$, there is finer str. .

– consider this setup in $\overline{\mathbb{F}}_p$.

grading on stalks given by Frobenius action on $\overline{\mathbb{F}}_p$

– q keeps track of this grading ...].