Lie 5/12/17 Dec.

Today : D-modules on $X = G/B$.

Action of $G$ on $X$. $\rightsquigarrow$ $\mathfrak{g} \to \mathrm{Vect}(X) = \Gamma(X, TX) \rightsquigarrow U\mathfrak{g} \xrightarrow{\varphi} D(X) = \Gamma(X, D_X)$

which commutes w/ action of $G$.

Thm (Beilinson-Bernstein, Kashiwara-Brylinski).

① $\varphi$ gives isom $U\mathfrak{g}/\ker\chi_0 \xrightarrow{\sim} D(X)$, where $\chi_0 : Z(U\mathfrak{g}) \to \mathbb{C}$ is central character.

$z|_{M(0)} = \chi_0(z)$

② $X$ is D-affine, i.e.

$\Gamma : \mathrm{Mod}_{qc}(D_X) \longmapsto \mathrm{Mod}(D(X))$ is equiv. of categories.

(sheaf of mod. (don't even need holonomic, coherent)) just mod. ← e.g. "$\mathbb{P}^1$ has enough global diff operators"

Cor. : $\mathrm{Rep}^n(\mathfrak{g}, \chi_0) \simeq \mathrm{Mod}_{qc}(D_X)$.

$\mathrm{Rep}^n(\mathfrak{g}, \chi_0)$ = Reps of $\mathfrak{g}$ on which $Z(U\mathfrak{g})$ acts by $\chi_0$

If want to consider other central characters, need to consider "twisted $D_X$",
→ instead of $D_X$ acting on $\mathcal{O}_X$,
twisted $D_X$ act on sections of line bundle.

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Pf of part 1 :

Consider $\mathrm{gr}\,\varphi : \mathrm{gr}(U\mathfrak{g}) \to \mathrm{gr}(D(X))$.

$\mathrm{gr}(U\mathfrak{g}) = S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$, $\mathrm{gr}(D(X)) \cong \mathbb{C}[T^*X]$.

Then $\mathrm{gr}\,\varphi = \pi^*$ for some $\pi : T^*X \to \mathfrak{g}$.

Lemma $\pi : T^*X \to \mathfrak{g}^*$ (is moment map of symplectic geometry/Hamiltonians).

is given by : $T^*X = \{(x \in X, v \in T^*_x X)\}$ $\quad X = G/B = \{B' \subset G\}$, $B'$ Borel.

$T_{B'}X = \mathfrak{g}/\mathfrak{b}'$ $\quad T^*_{B'}X = (\mathfrak{g}/\mathfrak{b}')^* \underset{\text{Killing}}{\simeq} (\mathfrak{b}')^{\perp}_{\mathfrak{g}} = \mathfrak{n}' = [\mathfrak{b}', \mathfrak{b}'] \subset \mathfrak{b}'$.

$= \{(\mathfrak{b}', a \in \mathfrak{n}')\}$.

$\pi(\mathfrak{b}', a) = a$ (iden. $\mathfrak{g}^* \underset{\text{Killing}}{\simeq} \mathfrak{g}$)

Thus, $\mathrm{im}\,\pi = \mathcal{N} = \{a \in \mathfrak{g} \mid a \text{ is nilp.}\}$.

(1) $\mathcal{N} = \{a \in \mathfrak{g}^* \mid s(a) = 0 \quad \forall s \in (S_{\mathfrak{g}})_+^G$ ~~except 1~~ $\}$ — from deg $\geq 1$. (want to exclude 1).

Note: (2) $\mathcal{N}$, the nilpotent cone, is not smooth.

(Kostant) e.g. $\mathfrak{sl}_2$: $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ — $\mathcal{N} = \{\det A = 0\} = \{a^2 + bc = 0\}$

(3) For dense open subset $\mathcal{N}_0 \subset \mathcal{N}$ of "regular nilpotent" elem., $= \{x_1^2 + x_2^2 + x_3^2 = 0\}$.

$\pi : \pi^{-1}(\mathcal{N}_0) \longrightarrow \mathcal{N}_0$ is an isom.

i.e. for most nilpotent, there is unique borel $\mathfrak{b} \ni n$.

e.g. in $\mathfrak{sl}_n$, $\mathcal{N}_0$ = rank $n-1$.

— there is unique flag that $n$ "shifts by 1".

$T^*X \xrightarrow{\pi} \mathcal{N}$ Thus $T^*X, \pi$ is a resolution of singularities (Springer resolution). — $\mathcal{N}$ is normal

~~$\mathbb{C}[\mathcal{N}] = \mathbb{C}[T^*X]$~~.

e.g. $\mathfrak{sl}_2$. $T^*\mathbb{P}^1 \longrightarrow$  $\mathcal{N} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$.

$\pi^{-1}(0) = \mathbb{P}^1 = \left(\text{zero sect. of } T^*\mathbb{P}^1\right)$.

$\mathbb{C}[\mathcal{N}] = \mathbb{C}[T^*X]$.

(is part of Kostant's results)

Thus, $\ker^{gr}_{\Lambda} \varphi = \{ f \mid f|_{\mathcal{N}} = 0\} = (S_{\mathfrak{g}})_+^G$

$\operatorname{im}^{gr}_{\Lambda} \varphi = \pi^* \mathbb{C}[T^*X] = \pi^*\mathbb{C}[\mathcal{N}]$.

Back to original question $\varphi : U_{\mathfrak{g}} / \ker \chi_0 \longrightarrow D(X)$. — we now have surj.

Thm

Prop. (1) $\varphi|_{\ker \chi_0} = 0$.

(2) $\mathrm{gr}(\ker \chi_0) = (S\mathfrak{g})^G_+$.

moreover, $\ker \chi_0 = Z(U\mathfrak{g}) \cap (U\mathfrak{g})_+ = (U\mathfrak{g})^G \cap (U\mathfrak{g})_+$

Pf (1): $\varphi(Z(U\mathfrak{g})) \subset D(X)^G$

and $D(X)^G$ is one-dim

(b/c $\mathrm{gr}(D(X)^G) = \mathbb{C}[T^*X]^G = \mathbb{C}[N]^G = \mathbb{C}$

so easy to see $\varphi(z) = \chi_0(z)$. — $N_0$ is the one-orbit & dense.

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$D(\mathbb{P}^1) = \langle 1,\ e = \partial_x,\ h = 2x\partial_x,\ f = -x^2\partial_x \rangle$.

$= \langle 1,\ x_1\partial_2,\ x_1\partial_2 - x_2\partial_1,\ x_2\partial_1 \rangle$.

$\mathfrak{sl}_2$: $Z(U\mathfrak{sl}_2) = \langle ef + fe + \frac{1}{2}h^2 \rangle$, $\varphi|_{\mathbb{C}\cdot L(0)} = 0$.

$U\mathfrak{sl}_2 / \ker \chi_0 = \langle e, f, h \mid [h,e] = 2e, \vdots, C = 0 \rangle$.

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Next task: $\mathrm{Rep}(\mathfrak{g}, \chi_0) \simeq \mathrm{Mod}_{q.c.}(D_X)$

$\mathcal{O} \subset \mathrm{Rep}(\mathfrak{g},\chi_0)$ $\longrightarrow$ $? \subset \mathrm{Mod}_{q.c.}(D_X)$ ~~various geometric constructions~~

$? \longleftarrow$ geometric constructions