Lie
30/11/17

Last time: ① Duality $\mathbb{D}: \cancel{\mathrm{Mod}}(D_X) \longrightarrow \cancel{\mathrm{Mod}}(D_X^{op})$.

$$D^b_h(D_X) \longrightarrow D^b_h(D_X).$$

In fact, ~~for~~ for holonomic D-module, M,

$\mathbb{D}M$ is again holonomic D-module (not complex).

$$\begin{array}{ccc} D^b_h(D_X) & \longrightarrow & D^b_h(D_X) \\ \cup & & \\ \mathrm{Mod}_h(D_X) & \longrightarrow & \mathrm{Mod}_h(D_X) \end{array}$$

$\mathbb{D}(\mathcal{O}_X) = \mathcal{O}_X$

$\mathbb{D}(\mathbb{C}[x,x^{-1}]) = D_X\,\theta \quad x\partial\cdot\theta = 0$.

② $f: X \longrightarrow Y$

$f^!: D(\mathcal{O}_Y) \longrightarrow D(\mathcal{O}_X)$. $f^! = Lf^*[\dim X - \dim Y]$.

f^* usual inverse image for $\mathcal{O}$-modules.

$f^* D_Y = D_{X\to Y}$, $(D_X, f^{-1}D_Y)$ bimodule.

e.g. $X \hookrightarrow Y$ closed. $X = \{y_{k+1} = \dots = y_n = 0\}$.

$D_{X\to Y} = D_X \otimes \mathbb{C}[\partial_{k+1},\dots\partial_n]$.

Today

③ Direct image $f: X \longrightarrow Y$.

f_* (or $\int_f$): $D(D_X) \longrightarrow D(D_Y)$.

For right D-modules, it is $\int_f \cancel{f_*} M = Rf_*\big(\underbrace{M \overset{L}{\otimes}_{D_X} D_{X\to Y}}_{\text{sheaf on }X}\big)$ $\quad \int_f M = Rf_*\big(M \overset{L}{\otimes}_{D_X} D_{X\to Y}\big)$

} left $\simeq$ right.

left D-mod.

usual direct image of sheaves.

right module over $f^{-1}D_Y$.

e.g. ① $X \hookrightarrow Y$ closed embedding.

$\int_f \mathcal{O}_X = \delta_X = \mathcal{O}_X \otimes \mathbb{C}[\partial_{k+1},\dots,\partial_n]$.

| e.g. $\mathbb{C} \downarrow \; \cdot \xrightarrow{i_0} \mathbb{C}$

$\int_{(i_0)} \mathbb{C} = \delta_0 = \mathbb{C}[\partial]\,\delta$.

② $\begin{array}{c} X \leftarrow F \\ \pi\downarrow \\ B \end{array}$ $\int_\pi \mathcal{O}_X$ = vector bundle on B ⟵ "integrate a top deg diff form on X along fibe, get top deg diff form on B"

fiber at $b \in B$ is $H^*(\pi^{-1}(b))$. (= complex of sheafs whose i^{th} hom is $H^i(\pi^{-1}(b))$

there is natural flat connection "Gauss-Manin connection").

(3) $j: U \hookrightarrow Y$ open embedding.

$$\int_j \mathcal{O}_U = Rj_*$$

e.g. $j_*\mathcal{O}_{\mathbb{C}^\times} = \mathcal{O}_{\mathbb{C}}(\infty\cdot 0)$: functions w/ pole at 0.

$j: \mathbb{C}^\times \hookrightarrow \mathbb{C}$ $\qquad \mathcal{O}_{\mathbb{C}}(\infty\cdot 0) = \mathbb{C}[x,x^{-1}]$

How does duality agree w/ f^*, $\int_f$.

Answer: (1) if f is proper,

then $\int_f \mathbb{D}M = \mathbb{D}\int_f M$

In general though, not true.

so define $f_! = \mathbb{D}\cdot\int_f\cdot\mathbb{D}$.

Eg. $\begin{array}{c} X \leftarrow F \\ \downarrow \\ B \end{array}$ $\qquad (f_!\mathcal{O}_X)_b = H^*_c(\pi^{-1}(b))$.

similarly, $f^\star := \mathbb{D} f^* \mathbb{D}$.

We also have from general theory canonical morphism

$$f_! M \longrightarrow \int_f M \ (= f_* M)$$

$D_x\theta \to \mathbb{C}[x,x^{-1}]$

$\theta \mapsto 1$

in

$0 \to \delta \to D_x\theta \to \mathbb{C}[x,x^{-1}] \to \delta \to 0$

$D_x\theta \to \mathbb{C}[x] \to \mathbb{C}[x,x^{-1}]$

$0 \to \mathbb{C}[x] \to 0$

regular $\quad f' = \frac{A(x)}{x} f \longrightarrow f \sim x^{\alpha}$.

$\updownarrow$

$D_x / D_x(\partial - \frac{A(x)}{x})$

D_x were invented to deal w/ diff eqn in algebraic setting

not regular:

$f' = \frac{A}{x^n} f$.

e.g. $f' = \frac{1}{x^2} f \qquad f = e^{-\frac{1}{x}} \rightsquigarrow$ really bad.

~~Important~~ example. (the rest eg are technicalities).

eg. $j: \mathbb{C}^\times \to \mathbb{C}$.

$j_*\mathcal{O}_{\mathbb{C}^\times} = \mathbb{C}[x,x^{-1}]$.

$j_!\mathcal{O}_{\mathbb{C}^\times} = \mathbb{D}(\mathbb{C}[x,x^{-1}]) = D_x\theta$

$0 \to \mathbb{C}[x] \to \mathbb{C}[x,x^{-1}] \to \mathbb{C}[\partial]\delta \to 0$

$0 \to \delta \to D_x\theta \to \mathbb{C}[x] \to 0$.

$\theta \longmapsto 1$

$\delta \longmapsto \partial\theta$

so to get $\mathcal{O}_{\mathbb{C}} = \mathbb{C}[x]$, it is neither j_* nor $j_!$, but $\mathrm{im}(j_! \to j_*)$.

This happens ~~a~~ in general:

Let $j : X \longrightarrow Y$ locally closed subvariety.

$M \in Mod_h(D_X)$ ~~(just~~ (just the ~~g~~ usual D-mod.). [even assume M ~ vect. bundle w/ flat connection].

Then: ① $\int_j M$, $j_! M$ are holonomic D_Y-mod (not cx, i.e. $0 \to . \to 0$).

(crucial ~~assumption~~ hypothesis: M is holonomic)

② Let $j_{!*} M = \mathrm{Im}(j_! M \longrightarrow \int_j M)$

= "intermediate extension". ~~Let~~ It is simple holonomic D_Y-mod.

③ Any simple holonomic D-module is of this form.

All of this should remain you of something ... "teaser" for next class ...

Compare w/ Rep of sl_2 ; $n \geq 0$, $n \in \mathbb{Z}$.

$$\begin{tikzcd} 0 \arrow[r] & M(-n-2) \arrow[r] & M(n) \arrow[r] \arrow[d] & L(n) \arrow[r] & 0 \\ 0 \arrow[r] & L(n) \arrow[r] & M^\vee(n) \arrow[r] & M(-n-2) \arrow[r] & 0 \end{tikzcd}$$

$L(n) = \mathrm{Im}(M(n) \to M^\vee(n))$.

All ~~that~~ $j_{!*} = \mathrm{Im}(j_! M \to \int_j M)$ is geometric counterpart of all this.