Lie. 28/11/17

Last time: $\mathcal{D}$-modules, $Mod(D_X) \simeq Mod(D_X^{op})$. } too large.

$\cup$

$Mod(D_X)$ - D-coherent

$\cup$

"good cat. of D-mod" ⟶ $Mod_h(D_X)$ (holonomic) — $\dim(ch(M)) = \dim X$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cap$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad T^*X$.

If M is holonomic,

then M is flat connection on open dense subvar $X_0 \subset X$.

Today: duality, functors.

(inv. image, direct image etc.).

However: all above work in derived cat. $D^b(D_X)$ — in the best cases, you just lose information: e.g. $\Gamma(X, V)$ for many vector bundles V is 0, → need to go to cohom. to get back more info.

$D^b(\mathcal{A})$: complexes of obj. of $\mathcal{A}$ up to q-isom.

q. isom are isom.

e.g. $P^\bullet \to A \to 0$ exact,

$\Rightarrow (P^\bullet \to 0) \simeq (0 \to A \to 0)$.

in derived

"don't make distinction between obj. & its resolution".

— note: this is not abelian cat.

q-isom ⟶ isom makes image, ker etc. bad.

— "exact seq." doesn't make sense, but replacement is exact triangle.

— looks weird, but useful for thinking about ideas.

e.g. If $F: \mathcal{A} \to \mathcal{B}$ left exact,

can define $R^iF(A) \in \mathcal{B}$.

(D^b: finitely many cohom.. "bounded"

D^+: bdd one way.)

Better: have functor $RF: D^b(\mathcal{A}) \to D^b(\mathcal{B})$. and

$\quad\quad\quad \cup$

$\quad\quad\quad A$ as $0 \to A \to 0$. $\quad R^iF(A) = H^i(RF(A))$.

$RF: A \mapsto$ inj res. $\longmapsto F$(inj. res.)

~~example~~ (1) Duality : (motivating example).

- A an algebra. Have duality functor

$$D : \mathrm{Mod}(A) \xrightarrow{D} \mathrm{Mod}(A^{T}).$$
$$M \longmapsto \mathrm{Hom}_A(M, A).$$

example where derived is useful →

e.g. $M = A/Aa$. $DM = \mathrm{Hom}_A(A/Aa, A)$

$= \{x \in A \mid ax = 0\}$

in case A has no 0-divisors, $DM = 0$ ...

not a very good functor...

Better: $M \simeq (0 \to A \xrightarrow{a} A \to 0)$ (still assuming no 0-div).

$$RDM \simeq (0 \leftarrow A \xleftarrow{a} A \leftarrow 0)$$

Thus $RD(M) \simeq M[-1]$. coker $H^1 = A/aA$ = right module.

	0	1
M	M	0
	0	-1
	0	2
M[-1]	M	1
	0	0

(1) Duality.

$$\mathbb{D} : D^b_h(D_X) \to D^b_h(D_X).$$

holonomic

$$\mathbb{D}(M^\bullet) = R\mathcal{H}om(M^\bullet, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[\dim X].$$

sheaf hom.

$\in D^b_h(D_X^{op})$

Examples : $X = \mathbb{C}$, $D = \mathbb{C}\langle x, \partial\rangle$.

$\mathbb{D}(\mathcal{O}_X) = \mathcal{O}_X$ $\qquad$ $\mathbb{D}(\delta_0) = \delta_0$.

$\mathbb{C}[x] = D_X/D_X\partial$. $\qquad$ $= D_X/D_X x$

A-right module dual of A/Aa is A/aA.

$$M = \mathbb{C}[x, x^{-1}] = D_X \cdot x^{-1} = D_X/\partial \cdot x.$$

$$\mathbb{D}M = D_X/\partial x D_X \rightsquigarrow D_X/D_X x\partial = \langle \theta \rangle$$

right $\qquad$ left $\qquad$ $x\partial\theta = 0$.

$\partial\theta = \delta$

$\theta = \int \delta$ ...

Thm ① If $M \in \mathrm{Mod}_h(D_X)$,

then $\mathbb{D}M \in \mathrm{Mod}_h(D_X)$ (i.e. $H^i(\mathbb{D}M) = 0$ $i \neq 0$, $H^0(\mathbb{D}M) \in \mathrm{Mod}_h(D_X)$.)

② $\mathbb{D}^2 = id$, ($\mathbb{D}$ is ~~contravariant~~.. $\mathbb{D}$ is antiequivalence of $\mathrm{Mod}_h(D_X)$).

(2) Inverse image

$f : X \longrightarrow Y$, $M \in \mathrm{Mod}(D_Y)$.

Claim: $f^*M = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(M)$ is D_X-mod.

$\theta(\varphi \otimes s) = (\theta\varphi) \otimes s + \varphi \otimes (f_*\theta) s$ — "push forward", first acting on stalk ...

$\varphi \in \mathcal{O}_X$

$s \in f^{-1}(M)$.

$\sum \theta \cdot (y_i \circ f) \cdot \partial_i s$.

, we will use:

$f^\dagger : D^b_h(Y) \longrightarrow D^b_h(X)$

$f^\dagger(M^\bullet) = {}^L f^* M^\bullet [\dim X - \dim Y]$. ($f^*$ is right exact).

Eg. $f^* D_Y =: D_{X\to Y}$ is a $(D_X, f^{-1}D_Y)$-bimodule.

then, for any D_Y-mod. M,

$f^*M = D_{X\to Y} \otimes_{f^{-1}(D_Y)} f^{-1}(M)$.

Example: $X \overset{i}{\hookrightarrow} Y$ closed ~~embd~~ embedding. (locally, coordinates in y's, $X = \{y_{k+1} = \dots = y_n = 0\}$.

$D_{X\to Y} = D_X \otimes \mathbb{C}[\partial_{k+1}, \dots, \partial_n]$.

$i^* \mathcal{O}_Y = \mathcal{O}_X$.

Thm. If $M^\bullet \in D^b_h(X)$, then $f^\dagger(M^\bullet) \in D^b_h(D_X)$.

In particularly, for every pt. $x \in V$

$\dim_{\mathbb{C}} H^j(i_x^\dagger M^\bullet) < \infty$. (checking $\in D^b_h \Leftrightarrow$ check at points)

$i_x : pt \hookrightarrow X$ — v.s. (of a cx of v.s.)