Lie 21/11/17.

Idea: ~~saw~~ instead of equiv bundle on G/B,

look for D-modules on G/B $= \mathrm{Mod}(D_{G/B})$. because action of G on G/B $\rightsquigarrow$ $Ug \longrightarrow D_{G/B}$.

Last time D-module: · flat connections eg. $\mathcal{O}_X$

· other eg. δ-functions.

· right ~~D~~ D-mod. : Ω_X.

$\rightsquigarrow$

$$\mathrm{Mod}(\mathcal{D}) \longleftrightarrow \mathrm{Mod}(\mathcal{D}^{op}).$$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^{top}$$

~~Wat~~ What ~~is~~ are good modules over $\mathcal{D}$?

(1) ~~All~~ All $\mathcal{D}$-modules will be quasi-coherent over $\mathcal{O}$

(2) Thm: A $\mathcal{D}$-mod. M is coherent over $\mathcal{O}$

$\iff$ M is locally free over $\mathcal{O}$, i.e. it is a flat connections.

Note: $\mathcal{D}$ itself is not $\mathcal{O}$-coherent. (x is nilpotent on δ-function) $\to$ not f.g.

(3) $\mathrm{Mod}_c(\mathcal{D})$: q.c. over $\mathcal{O}$, finitely gen. over $\mathcal{D}$. ("coherent $\mathcal{D}$-modules").

"counterpart of f.g. repns".

eg (1) $\mathcal{D}$ itself.

(2) Any quotient of $\mathcal{D}$, eg. $\langle \delta \rangle = D_X / D_X \cdot x$ on $\mathbb{C}$.

(4)

(4) Even better class of $\mathcal{D}$-modules :

~~kt~~ holonomic $\mathcal{D}$-modules.

Define holonomic $\mathcal{D}$-mod., but first some preliminaries ...

(1) locally : $\mathcal{D}(U)$ is filtered ~~by~~ algebra :

$$F^p\mathcal{D} = \{ \text{diff op of order} \le p \}.$$

$$\begin{array}{ccccccc} 0 \subset & F^0\mathcal{D} & \subset & F^1\mathcal{D} & \subset & F^2\mathcal{D} & \subset \cdots \\ & \| & & \| & & & \\ & \mathcal{O} & & \mathcal{O} + \mathrm{Vect} & & & \end{array}$$

$$Gr(\mathcal{D}) = \bigoplus_{p \ge 0} F^p/F^{p-1} \quad (\text{locally } \mathbb{C}[x_i, \partial_i])$$

$$= \mathcal{O}(T^*U)$$

In fact, ~~have fi~~ filtrations over different U's agree,

so have filtration of sheaves.

& can have ~~graded~~ sheaf of graded algebras

{ For $\Psi \in F^p\mathcal{D}$, its image $\sigma(\Psi)$ in $Gr^p(\mathcal{D})$ is called symbol of Ψ }.

(2) Let M be coherent $\mathcal{D}$-mod.

A good filtration of M is a filtration

$$\cdots \subset F^pM \subset F^{p+1}M \subset \cdots$$

which is compatible w/ filtration on $\mathcal{D}$.

~~Thm: Every coherent $\mathcal{D}$-module admits such a filtration~~

<u>Thm</u>: Every coherent $\mathcal{D}$-module admits such a filtration.

Eg. (1) : $\mathcal{O}_X = F^0\mathcal{O}_X = F^1\mathcal{O}_X = \dots$

(2) : δ-function $= D/D\cdot x = \mathbb{C}[\partial_x]\cdot\delta = \langle \delta, x\partial_x\cdot\delta, \dots \rangle$

$F^k = \langle \delta, \partial\cdot\delta, \dots, \partial^k\cdot\delta \rangle$.

In this case, $Gr(M) = \bigoplus F^pM/F^{p-1}M$.

is a module over $Gr(\mathcal{D}) \simeq \mathcal{O}(T^*X)$.

i.e. a coherent sheaf on T^*X.

Eg. For $M = \mathcal{O}_X$, $Gr(M) = \mathbb{C}[x_i, \partial_i]/\langle \partial_i \rangle$ ← think of $\mathbb{C}[x,y]/(y)$.

$= \mathcal{O}_{X \hookrightarrow T^*X}$ (pushforward of

$\left(= i_*\mathcal{O}_X,\ i: X \overset{0}{\hookrightarrow} T^*X\right)$..

For $M = \delta$-fn. at $0 \in \mathbb{C}$,

$Gr(M) = \mathbb{C}[x, \partial_x]/(x)$

$= \mathcal{O}_{\pi^{-1}(0)}$

For $M = D$: $Gr(M) = \mathcal{O}(T^*X)$.

(3) Let M be coherent $\mathcal{D}$-module. We define <u>characteristic variety</u> (has roots/relations to PDE stuff.)

$ch(M) \subset T^*X$ by $ch(M) = supp(Gr(M))$. (as alg. var. ~~not~~ kill non-reducedness. not scheme.)

Thm : For any coherent $\mathcal{D}$-mod., $\dim(ch(M)) \geq n = \dim X$.
of each component.

((and if $\dim = n$, then $ch(M)$ is Lagrangian).

[a rather non-trivial thm]

——— prelim done ———

Defn: A coherent $\mathcal{D}$-mod. is called holonomic if $\dim(ch(M)) = \dim X$.

Eg. $M = \mathcal{O}_X$.

and in general, any vect. bundle w/ flat connection.

M = δ-fn.

In a sense, these are "smallest" $\mathcal{D}$-mod, in terms of support...

Just saying coherent / $\mathcal{O}$ "singles out the x direction", wouldn't include the δ-fn.

Holonomic is right way to generalize while keeping supp small...

So, we will be working w/ holonomic $\mathcal{D}$-mod.

{later will add regular singularities}

Do we know all $\mathcal{D}$-mods? First step in clarifying:

Are they all of form $\mathcal{O}_X$, δ-fn...?

Thm : Let M be hol. $\mathcal{D}$-mod on X.

Then there exists open $U \subset X$, s.t. $M|_U$ = flat connection.

(e.g. δ-fn : take $U = X \setminus \pi^{-1}(0)$!)

Idea: there's some stratification of X, so that as codim $\uparrow$, supp become more "singular"

more like δ-fn