

Lie 16/11/17

Last time: Borel-Weil-Bott.

$$H^p(G/B; L_\lambda) = \begin{cases} \mathbb{P} L(\omega(-N)) & l(\omega) = p \\ 0 & \text{otherwise} \end{cases}$$

Goal: get all modules in  $\mathbb{O}$ , eg.  $M(\lambda), M^\vee(\lambda), L(\lambda), \dots$  ,  $\infty$ -dim...

- no hope of getting in ~~an~~ <sup>some  $G$ -equivariant</sup> similar way, because the  $G$  action <sup>of on these</sup> doesn't integrate.

- maybe  $\mathbb{V}$  integrates to  $B$  action.  
action on  $(B \backslash G/B)$  is finite-dim.

Action of  $G$  on  $X = G/B$  gives  $g \rightarrow \text{Vect}(X)$ .  $U(\text{Vect}(X))$  locally.

want to consider local situation,  $Ug \rightarrow D(X) =$  differential operators on  $X$

so... sheaf version of  $D(X)$  ...

(eg.  $\mathbb{C}$  on  $\mathbb{P}^1$   
 $e = x^2 \partial_x$   
 $f = -\partial_x$ )

Def: Let  $X$  be smooth alg/analytic var.

$D_X =$  sheaf of diff operators on  $X$ .

locally,  $D_X(U) = \left\{ \sum c_i(x) \partial_{x_1}^{a_1} \dots \partial_{x_n}^{a_n} \right\}$  (either in alg/analytic setting).  
 $\mathbb{O}(U)$

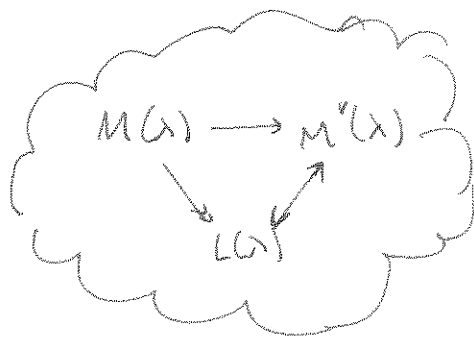
A  $D_X$ -module is a sheaf of abelian gps on  $X$ ,

st.  $M(U)$  is module over  $D_X(U)$ . ie.

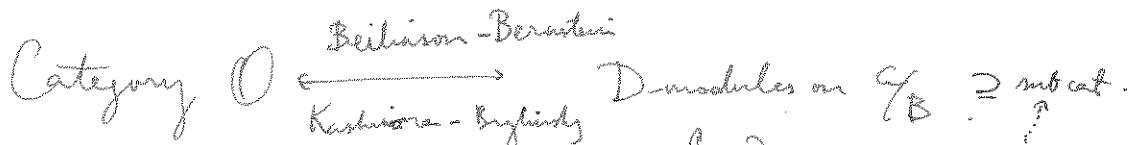
$M(U) \otimes_{D_X} M \rightarrow M$ . (in particular,  $M$  is  $\mathbb{C}$ -mod.)

Ideas: Cartan - Lusztig 1970's.

Hope: produce  $D_X$ -module that whose space of sections would give reprs of  $\mathfrak{g}$ .



Overview of programs (historically).



sort of a local system w/ singularities ... eg. the  $P^1$ ,  $S^1$  case

still rather hard to compute...

Riemann-Hilbert correspondence (a equivalence of suitably defined categories).

Constructible sheaves, perverse sheaves (Goresky, MacPherson).

Example of  $D_X$  mod.:

$M =$  sections of vector bundle w/ flat connection.

$$\nabla_X : M(U) \rightarrow M(U)$$

$$[\nabla_X, \nabla_Y] = \nabla_{[X, Y]} \rightarrow \text{so isoth } \nabla_X \text{ is } D_X \text{ action}$$

or, look at flat sections, gives trivialization of sheaf.

locally eg. some trivial sheaf on  $\mathbb{C} \setminus \{0\}$ , can always extend to 0?

no: depends on monodromy around 0.

- if monodromy is trivial, then can.

- if not, extend by 0 (sections at 0 are 0).

- still "locally constant", but the dimension is jumps.

$M(U), M'(U), L(U)$  is obtained by extending the sheaf on the open Borel cell in different ways (\*, !, !\*).

given by Cartan & Lusztig

$$\begin{aligned} M(w; \lambda) &= L(w; \lambda) \\ &= P_{w, \lambda}(q=1) \end{aligned}$$

( $\mathbb{C} = \mathbb{F}_q$ ).

$$\chi(G) = \left( \begin{array}{c} \# \text{ pt in } G \\ q=1 \end{array} \right)$$

Reference: Hotta, Takeuchi, Tanisaki.

D-modules, perverse sheaves & repres. theory.

Diffi Difficulty in the program: need to pass to derived category.

House some examples of  $D_x$ -modules.

↑  
confusing at first,  
but might be more natural  
than language of abelian  
cat.

(1)  $\delta$ -functions:  $X = \mathbb{C}$ .

$$D_x = \mathbb{C}[x, \partial_x] / \partial_x^2 x - x \partial_x = 1.$$

(a)  $M = \langle \begin{smallmatrix} 1 \\ e \end{smallmatrix} \mid \partial_x e = 0 \rangle$  (there is sheaf version).

$$= D_x / \begin{smallmatrix} D_x \cdot \partial_x \\ D_x \cdot \partial_x \end{smallmatrix} \simeq \mathbb{C}[x] = \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

$(\mathbb{C}[x] \cdot e = \mathbb{C}[x] \cdot 1)$ .

(b)  $N = \langle \begin{smallmatrix} \delta \\ 0 \end{smallmatrix} \mid x \cdot \delta = 0 \rangle$

$$= D_x / \begin{smallmatrix} D_x \cdot \partial_x x \\ D_x \cdot \partial_x x \end{smallmatrix} \simeq \mathbb{C}[\partial_x] \cdot \delta.$$

↖ as  $\mathcal{O}_x$ -mod., very different.  
↙ as  $D_x$ -mod., equally good.

\* sheaf version:  $N = \Gamma(\mathbb{C}, \mathcal{N})$

$\mathcal{N}$ : sheaf supported at  $x=0$ .

Right  $D_x$ -modules.  $M \otimes_{D_x} \rightarrow M$ .

Examples is there right action on functions? we no, but have on distributions

space of  
distributions is nasty,  
analytic difficulties...

$$\mathcal{O}(x)^* = \{ \psi: \mathcal{O}(x) \rightarrow \mathbb{C} \}.$$

$$\langle \psi D, f \rangle := \langle \psi, Df \rangle.$$

simpler version:  $\psi \in \Omega_x^{\dim X}$  (pair of  $f$  and  $\int$ ).

Lemma: there is a well-defined right action of  $D_x$  on  $\Omega_x^{\text{top}}$ ,

locally defined by  $\left\{ \begin{array}{l} (f(x) \cdot dx^I) \cdot \partial_{x_j} = -( \partial_{x_j} f ) \cdot dx^I \end{array} \right.$  (integration by parts).

[check].

note: the pairing only works if  $X$  cpt...  
this is the correct version.

Then:  $\text{Mod}(D_X) \xrightarrow[\text{equivalence of categories}]{\sim} \text{Mod}(D_X^{\text{op}})$   
 left  $D_X$ -modules

$$F \xrightarrow{\quad} \Omega_X^{\text{top}} \otimes_{D_X} F$$

Exercise: write explicitly right action of  $D_X$  on  $\Omega_X^{\text{top}} \otimes F$ .

Next time: in AG, question to ask: pullback of sheaves  
 direct image

is the push functors for  $D_X$ -modules?

$\implies$  if so, can we extend  $D_X$ -mod to from subvariety (not nec. closed) to whole variety

answer is yes.

[refresh direct image functors, inverse image functors etc., i.e. open Hartshorne]