Lie 16/11/17

Last time: Borel-Weil-Bott.

$$H^p(G/B; L_\lambda) = \begin{cases} L(w.(-\lambda)) \; (w \in P_+), & l(w) = p \\ 0 & \text{otherwise} \end{cases}$$

Goal: get all modules in $\mathcal{O}$, e.g. $M(\lambda)$, $M^\vee(\lambda)$, $L(\lambda)$, ... (∞-dim...)

- no hope of getting in similar way, because the $\mathfrak{g}$ action on these doesn't integrate. (some G-equivariant stuff.)
- $\mathfrak{b}$ action integrates to B action. ($\mathfrak{b} \cdot v$ is fin-dim.)

Action of G on $X = G/B$ gives $\mathfrak{g} \to \mathrm{Vect}(X)$ (acts on sections of vect. bundle). $U(\mathrm{Vect}(X))$ locally.

$U\mathfrak{g} \to D(X)$ = differential operators on X.

want to consider local situation, so ... sheaf version of $D(X)$ ...

(e.g. Sl_2 on $\mathbb{P}^1$: $e = x^2\partial_x$, $h = 2x\partial_x$, $f = -\partial_x$.)

Def: Let X be smooth alg/analytic var. $/\mathbb{C}$.

D_X = sheaf of diff operators on X.

locally, $D_X(U) = \{ \sum c_\alpha(x) \partial^\alpha \}$ (either in alg / analytic setting), $c_\alpha(x) \in \mathcal{O}(U)$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

- A D_X-module is a sheaf M of abelian gps on X, s.t. $M(U)$ is module over $D_X(U)$. i.e.

$D_X \otimes M \to M$. (in particular, M is $\mathcal{O}_X$-mod.)

Ideas: Cartan - ~~Lus~~ Lusztig 1970's.

Hope: produce D_X-module ~~that~~ whose space of sections would give rep'ns of $\mathfrak{g}$.

$$\begin{tikzcd} M(\lambda) \arrow[r] \arrow[dr] & M^\vee(\lambda) \arrow[d, leftrightarrow] \\ & L(\lambda) \end{tikzcd}$$

~~some~~ Overview of program (historically).

Category $\mathcal{O}$ $\xleftrightarrow[\text{Kashiwara - Brylinski}]{\text{Beilinson - Bernstein}}$ D-modules on G/B $\supseteq$ sub cat.

still rather hard to compute...

Rie~~mann~~-Hilbert correspondence.

(an equivalence of suitably defined categories).

sort of a local system w/ singularities ... eg. the $\mathbb{P}^1$, SL_2 case

Constructible sheaves, $\cup$ ~~perf~~ perverse sheaves (Goresky, MacPherson)

essentially locally trivially $\leadsto$ locally constant sheaf on Bruhat cells.

passing to $\mathbb{F}_q$: combinatorics: Hecke algebras.

Example of ~~D_X~~ D_X-mod.:

M = sections of vector bundle w/ flat connection.

$\nabla_\xi : M(U) \longrightarrow M(U)$.

$[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$ $\leadsto$ so ~~is~~ $\nabla_\cdot$ is D_X action

Or, ~~s~~ look at flat sections, gives trivialization of sheaf. local.

Answer to $[M(w\cdot\lambda) : L(w'\cdot\lambda)] = P_{w,w'}(q=1)$ — KL-polynomial. given by Cartan & Lusztig

$(\mathbb{C} = \mathbb{F}_1)$.

χ (on Gr) = $\begin{pmatrix} \#\text{pt in } Gr_{\mathbb{F}_q} \\ q=1 \end{pmatrix}$.

eg. some locally trivial sheaf on $\mathbb{C}\setminus 0$, can always extend to 0?

no: depends on monodromy around 0.

— if monodromy is ~~no~~ trivial, then can.

— if not, extend by 0 (stalk ~~sections~~ at 0 ~~is~~ 0).

— still "locally constant", but the dimension ~~is~~ jumps.

— $M(\lambda)$, $M^\vee(\lambda)$, $L(\lambda)$ is obtained by ~~ext~~ extending the sheaf ~~on~~ on the open Bruhat cell in different ways ($*$, $!$, $!*$.

Reference: Hotta, Takeuchi, Tanisaki: D-modules, perverse sheaves & represen. theory.

~~Diffe~~ Difficulty in the program: need to pass to derived category.

confusing at first, but might be more natural than language of abelian cat.

~~Some~~ Some examples of D_X-modules.

(1) δ-functions: $X = \mathbb{C}$.

$$D_X = \mathbb{C}[x, \partial_x] \Big/ \partial_x x - x\partial_x = 1.$$

(a): $M = \langle e \mid \partial_x \cdot e = 0 \rangle$ (there is sheaf version).

$$= D_X \big/ D_X \cdot \partial_x \simeq \mathbb{C}[x] = \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

$(\mathbb{C}[x]\cdot e = \mathbb{C}[x]\cdot 1)$.

(b) $N = \langle \delta \mid x \cdot \delta = 0 \rangle$

$$= D_X \big/ D_X \cdot x \simeq \mathbb{C}[\partial_x] \cdot \delta.$$

as $\mathcal{O}_X$-mod., very different.
as D_X-mod, equally good.

sheaf version: $N = \Gamma(\mathbb{C}, \mathcal{N})$

$\mathcal{N}$: sheaf supported at $x = 0$.

Right D_X-modules. $M \otimes D_X \longrightarrow M$.

Examples is there right action on functions? no, but have on distributions

$$\mathcal{O}(X)^* = \{\varphi : \mathcal{O}(X) \to \mathbb{C}\}.$$

space of distribution is nasty, analytic difficulties...

$$\langle \varphi D, f \rangle := \langle \varphi, Df \rangle.$$

simpler version: $\varphi \in \Omega_X^{\dim X}$ (pair of f and $\int$).

Lemma: there is a well-defined right action of D_X on Ω_X^{top},

locally defined by $(f(x) \cdot dx^I) \cdot \partial_{x_j} = -(\partial_{x_j} f) \cdot dx^I$ (integration by parts).

[check].

note: the pairing only works if X cpt... this is the correct version.

Thm: $\mathrm{Mod}(D_X) \xrightarrow{\sim} \mathrm{Mod}(D_X^{op})$.

left D_X-modules; equivalence of categories.

$$\mathcal{F} \longmapsto \Omega_X^{top} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Exercise: write explicitly right action of D_X on $\Omega_X^{top} \otimes \mathcal{F}$.

Next time: in AG, natural to ask: pullback of sheaves
direct image
⋮

is there such functors for D_X-modules?

⇝ if so, can ~~get~~ extend D_X-modules from subvariety (not nec. closed) to whole variety.

Answer is yes.

[refresh direct image functors,
inverse image functors etc., i.e. open Hartshorne.]