Lie 14/11/17.

Borel-Weil (+ Bott).

~~Recall~~ Last time: $X = G/B = K/T_K$. (e.g. $SL(n)/\text{upper tri} = SU(n)/\text{diagonal}$.)

L_λ - line bundle. $\lambda \in P$

BW Thm: $\Gamma(X, L_\lambda) = \begin{cases} 0 & \lambda \in P_+ - P_+ \\ L(w_0\lambda) & \lambda \in -P_+ \end{cases}$

$L(w_0\lambda)$ ← lowest weight.

Pf: $C^\infty(X, L_\lambda) = \hat{\bigoplus}_{\mu \in P_+} (L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda)^{T_K}$
$= \hat{\bigoplus}_{\mu \in P_+} L(\mu)^* \otimes (L(\mu))_{-\lambda}$.

$\Gamma = \{s : \bar\partial s = 0\}$
$= \{s : n_+ s = 0\}$
$= \bigoplus L(\mu)^* \otimes (L(\mu)_{-\lambda})^{n_+} = L(-\lambda)^*$ if $-\lambda \in P_+$

Now add Bott: $H^i(X, L_\lambda) = ?$

Recall: Dolbeaux cohomology:

E – hol. vec. bundle on X.

~~$H^{p,q} = \{ \Omega^{p,q} \otimes E$~~

$H^{p,q}(\Omega^{\bullet,\bullet} \otimes E, \bar\partial)$.

$\bar\partial : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$.

$dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar z_{j_1} \wedge \ldots \wedge d\bar z_{j_q}$ — C^∞. forms.

$$\boxed{H^i(X, E) = H^{0,i}_{\bar\partial}(E)}$$

Thus, $H^\bullet(X, L_\lambda) = H^\bullet(\Omega^{0,\bullet} \otimes L_\lambda, \bar\partial)$. $\Omega^{0,\bullet} = C^\infty \otimes \Lambda^\bullet (T^{0,1})^*$

For flag variety, $T^{0,1} = n_+$ [check ..].

so $\Omega^{0,i} \otimes L_\lambda = \Lambda^i n_+^* \otimes C^\infty(L_\lambda)$.

& so $H^i(X, L_\lambda) = H^i(\Lambda^\bullet n_+^* \otimes (L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda)^{T_K}, \bar\partial)$

T_K normalizing action of n_+ commutes w/ $\bar\partial$

$= H^i(\Lambda^\bullet n_+^* \otimes L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda, \bar\partial)^{T_K}$

$= \bigoplus_\mu L(\mu)^* \otimes H^i(n_+, L(\mu))_{-\lambda}$.

[Recall: $H^i(\mathfrak{a}, V) = \mathrm{Ext}^i_{\mathfrak{a}}(\mathbb{C}, V)$

computed from ~~resolution of $\mathbb{C}$~~.

$U\mathfrak{a} \otimes \Lambda^i \mathfrak{a}$.

need to see that differentials

$\bar{\partial}$ on $\Lambda^\bullet \mathfrak{n}_+^* \otimes (L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda)$.

d in $\Lambda^\bullet \mathfrak{a}^* \otimes V$ (from abstract nonsense) $\longrightarrow$

are same. [exercise].

so $H^i(\mathfrak{a}, V) = H^i(\mathrm{Hom}_{\mathfrak{a}}(U\mathfrak{a} \otimes \Lambda^\bullet \mathfrak{a}, V))$

$= H^i(\mathrm{Hom}_{\mathbb{C}}(\Lambda^\bullet \mathfrak{a}, V))$

"bar resolution" $= H^i(\Lambda^\bullet \mathfrak{a}^* \otimes V)$. free

[exercise: get the differential here]

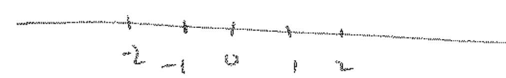
$$H^i(X, L_\lambda) = \bigoplus_\mu L(\mu)^* \otimes H^i(\mathfrak{n}_+, L(\mu))_{-\lambda}.$$

<u>Thm.</u> (Konstant) $\quad H^i(\mathfrak{n}_+, L(\mu)) = \bigoplus_{\substack{w \in W \\ \ell(w) = i}} \mathbb{C}_{w \cdot \mu}$.

<u>Pf</u>: later.

$$\Rightarrow H^i(X, L_\lambda) = \begin{cases} L(\mu)^*, & -\lambda = w \cdot \mu,\ \ell(w) = i,\ \mu \in P_+. \\ 0, & \text{else}. \end{cases}$$

<u>eg</u>: ~~$\mathfrak{gl}_2$~~ $\mathfrak{sl}_2$, $H^i(\mathbb{P}^1, L_\lambda) = \{$



$-\lambda \geq 0$: $H^0 = L(-\lambda)^* \cong L(-\lambda)$
$H^1 = 0$ $\qquad \mathcal{O}(n \geq 0)$

$-\lambda \leq -2$: $H^0 = 0$
$H^1 = L(\lambda - 2)$. $\qquad$ [direct check: $L_\lambda \cong \mathcal{O}(-\lambda)$.
$\mathcal{O}(n \leq -2)$

$-\lambda = -1$: $H^0 = H^1 = 0$. $\qquad \mathcal{O}(n = -1)$.

{ If $-\lambda$ is on the shifted wall, then all cohomology vanishes.
Else, exactly one H^i is nonzero. }

Pf of Konstant's thm:

follows from BGG resolution

(Note: historically, Bott proved it before Konstant's thm using other methods)

$$\bigoplus_{\ell(w)=i} M(w.\lambda) \to \cdots \to L(\lambda) \to 0$$

Note: $M(w.\lambda)$ is free over $\mathfrak{n}_-$, so can use ~~resolution~~ this res. to compute

$$\mathrm{Ext}_{\mathfrak{n}_-}(L(\lambda), \mathbb{C}) = \mathrm{Ext}_{\mathfrak{n}_-}(\mathbb{C}, L(\lambda)^*)$$ (tensor w/ fin. dim module is exact...?)

$$\shortparallel$$

$$\bigoplus_{\ell(w)=i} \mathbb{C}_{-w.\lambda}$$ $\rightsquigarrow$ use aut of $\mathfrak{g}$ to turn $\mathfrak{n}_- \to \mathfrak{n}_+$, juggling signs .... $\square$.

Note: Pf is analytic in nature ...

Fully Algebraic geometry pf exists, but we won't go there...

~~Ess~~ Essentially, BGG, Bott, Konstant are equivalent.

Note:
Characters from Bott: compute (graded) Euler char of $L\lambda$ in different ways..
of $L(\lambda)$
from Bott, & from some action + consider fixed pt ..

Next time: ~~type~~ D-modules & perverse sheaves

eventually get the infinite dim reps, and whole of $\mathcal{O}$.