Lie 9/11/17

Recall: Borel-Weil Thm.

L_λ : line bundle on G/B. [note: there might be some issue w/ notation in previous lecture]

$L_\lambda = G \times_B \mathbb{C}_\lambda$, sections of L_λ = functions $f: G \to \mathbb{C}_\lambda$, st. $f(gb) = b^{-1} f(g)$.

Thm ① If $\lambda \in P_+$,

then $H^0(G/B, L_{-\lambda}) = L(-w_0(\lambda))$ = irr. f.d. w/ lowest wt. $-\lambda$

② Otherwise, $H^0(G/B, L_{-\lambda}) = 0$

Sketch of pf: $G/B = X_{w_0} \cup D_i$, construct section on X_{w_0}, check ~~pf~~ that it extends to whole of some $\mathbb{P}^1$ that is $\pitchfork D_i$.

open cell $= B\dot{w}_0 B \cong N$.

Today: alternative pf.

Let K = cpt form of G: $[K \subset G,\ Lie(G) = Lie(K) \otimes \mathbb{C}]$:

(Eg. $G = SL(n,\mathbb{C})$, $K = SU(n)$).

Thm: ① Such a cpt form always exists.

② ~~k = L~~ $k = Lie(K)$ is generated by

(for ss. G.) ih_α, $e_\alpha - f_\alpha$, $i(e_\alpha + f_\alpha)$. for suitable choice of root basis $e_\alpha, f_\alpha, \mathfrak{g}$ in $\mathfrak{g}$.

(to construct, first need to find $k \subset \mathfrak{g}$, st $K|_k$ is -ve defn. then somehow show it integrates to closed $\subset G$.).

Thus, we have

$$\begin{array}{ccc} G & \supset & T \\ \cup & & \cup \\ K & \supset & T_K \end{array} \qquad \begin{array}{c} \mathbb{C} \\ \\ \mathbb{R} \end{array}$$

Note: ~~K~~ Borel has no counterpart in K: there are no "upper triangular" in K, all $s \in K$ are diagonalizable.

Thm. $G/B \simeq K/T_K$. (isom of C^∞-mfld, also preserves action of K).

Example $G = SL_n$. $G/B = \{\text{flags in } \mathbb{C}^n\}$

$K/T_K = SU(n) / \text{diag. unitary matrices}$ — Gram-Schmidt

$= \{\text{ordered orthonormal bases in } \mathbb{C}^n\} / v_i \mapsto v_i e^{i\varphi_i}$, $\varphi_i \in \mathbb{R}$ — "phase change".

$= \left\{ \text{splitting } \mathbb{C}^n = W_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} W_n,\ \dim W_i = 1 \right\}$.

In general, use decomp.

$$G = K\underset{\exp(\mathfrak{h}_{\mathbb{R}}) = \langle h_\alpha \rangle}{A}N$$

(in $SL(n,\mathbb{C})$, A is $=\text{diag}(+ve, +ve, \dots, +ve)$):

$K = SU(n)$ = "rotation"

A = rescaling

N = linear combinations / column reduction

"is the Gram-Schmidt process".

$G/B = G/TN$

$= KAN/TN = KAN/T_K AN$

but $T = T_K \cdot A$, $= K/T_K$. $\square$

Cor: G/B is cpt.

Claim: The line bundle $L_\lambda = K \times_{T_\lambda} \mathbb{C}_\lambda$.

sections of L_λ = functions $f : K \to \mathbb{C}_\lambda$

$f(kt) = t^{-1} f(k)$.

$$C^\infty(G/B, L_\lambda) = C^\infty(K/T_K, L_\lambda).$$

We can easily describe $C^\infty(K/T_K, L_\lambda)$.

<u>Peter-Weyl</u> thm: $C^\infty(K) = \widehat{\bigoplus_{\lambda\in P_+}} L(\lambda)\otimes L(\lambda)^*$ as K-bimodules.

$$(g\mapsto \langle g\cdot v, \eta\rangle) \longleftrightarrow v \otimes \eta.$$

(more precisely, map $C^\infty(K) \leftarrow \bigoplus_{\lambda\in P^+} L(\lambda)\otimes L(\lambda)^*$ has dense image in $C^\infty(K)$. so take completion on the right by appropriate norm. (works for say $L^2(K)$)).

[note: K cpt is ~~goo~~ ~~important~~.
→ in Lie, C^∞ world, cpt is good
in alg. gp world, s.s. is good.]

$$C^\infty(K/T_K, L_\lambda) = \{ f: K\to\mathbb{C} \mid f(kt) = \lambda(t)^{-1}\cdot f(k)\}.$$

$$= \left(\widehat{\bigoplus_{\mu\in P_+}} L(\mu)^*\otimes L(\mu)\right)\Bigg|_{\text{wght } -\lambda \text{ in second factor}}.$$

[Check sign] →

$$= \widehat{\bigoplus_{\mu}} L(\mu)^*\otimes L(\mu)_{-\lambda}.$$

$\square$

But B-W thm is abt holomorphic ~~s~~ sections, K is not even complex ...
So how to detect whether f is "hol. on K/T_K"?

<u>Lemma</u> $H^0(G/B, L_\lambda^{\cancel{*}})$ = hol. sections of L_λ

$$= \{ f\in C^\infty(K/T_K, L_\lambda) \mid \mathfrak{n}_+ f = 0\}.$$

$$= \{ f\in C^\infty(K/T_K, L_\lambda) \mid \underbrace{f(kt) = \lambda^{-1}(t)\cdot f(k),\ \mathfrak{n}_+ f = 0}\}.$$

where for $z\in\mathfrak{g}$, $z = a+bi$, $a, b\in \cancel{k}$.

let $z\cdot f = a\cdot f + i b f$.
(from right action of K on itself)

eg. for $\mathfrak{sl}_2$:

$$e_\alpha = \underbrace{\tfrac{1}{2}(e_\alpha - f_\alpha)}_{a} - \tfrac{i}{2}\underbrace{(i(e_\alpha + f_\alpha))}_{b}.$$

essentially is Cauchy-Riemann eq.

Pete

Thus: $H^0(G/B, L_\lambda) = \left(\hat{\bigoplus} L(\mu)^* \otimes L(\mu)_{-\lambda}\right)^{n_+}$

$$= \hat{\bigoplus} L(\mu)^* \otimes L(\mu)_{-\lambda}^{n_+}$$

$$= \begin{cases} \mathbb{C}\cdot v_{*\mu} & \text{when } \mu = -\lambda \\ 0 & \text{otherwise} \end{cases}$$

so $H^0(G/B, L_\lambda) \cong \begin{cases} L(-\lambda)^* & \text{if } -\lambda \in P_+ . \\ 0 & \end{cases}$

Next time: $H^{>0} \ldots = ?$