Lie. 2/11/17. (Nov)

Recall:

G - s.s. alg gp w/ Lie $\mathfrak{g}$.

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ $\qquad$ $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ = \mathrm{Lie}(B)$ ← Borel.

$\mathfrak{h} = \mathrm{Lie}(T)$, $T \subset G$ max torus $\qquad$ $\mathfrak{n}_+ = \mathrm{Lie}(N)$, $N \simeq \mathbb{C}^{|R_+|}$

$X = G/B = \coprod_w \underbrace{BwB/B}_{X_w}$ $\qquad$ Each $X_w \simeq \mathbb{C}^{\ell(w)}$.

In particular, $X_{w_0} \simeq Bw_0 B/B = \cancel{B}\, w_0 B_- B/B = w_0 N_- B/B \simeq N_-$ alg var.

Today, Borel-Weil thm. (note: not Weyl.)

constructing reps of G geometrically.

G acts on $X = G/B$, so acts on fns on X, but its projective, so not interesting.

try ~~line~~ sections of ~~line~~ vector bundle. $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$.

— not enough data to define $G \circlearrowright \Gamma(X, L)$.

- fibres are not canonically identified.
- could say $g^* L$, but it's not quite the same...

Def*: A G-equivariant vector bundle is a vector bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ together w/ an action of G on the total space, which commutes w/ projection.

In particular, every $g \in G$ gives isom $g_*: L_x \xrightarrow{\sim} L_{g.x}$.

(eg. G-eq. vec. bundle over pt is just a repn of G).

Lemma: If G acts on X, and L is G-equiv $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$, then $\Gamma(X, L)$ is repn.

Thm: $\left(G\text{-equiv line bundles}/X \right) \longleftrightarrow \left(\begin{array}{c}\text{one-dim reps}\\ \text{of } B\end{array} \right) \longleftrightarrow \left(\overset{\text{"characters"}}{\lambda \in P} \right)$.

(assuming we take the 'largest' s.conn G).

$\wr\!\wr$

$B \supset N$ has to acts by 1 (since $N = [B,B]$).

so $B \subset \mathbb{C}$ is determined by action of $T \subset B$.

write $\mathcal{L}_\lambda$ for the l.b. corresp. to λ.

Pf: $(\longrightarrow)$ $\mathcal{L}$ equiv l.b., ~~for~~ look at V-fibre over $x_0 = 1 \cdot B \in G/B$.

so $\mathrm{Stab}(x_0) = B$ must act on V.

$(\longleftarrow)$ given $V \circlearrowleft B$,

$\mathcal{L} = G \times_B V = (g,v) / (g \cdot b, v) \sim (g, b \cdot v)$. "induction of reps".

Eg. $G = SL_2(\mathbb{C})$,

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

{More generally,

$(G \supset H$, rep of $H)$

$\downarrow$

equiv bundle on G/H.

$G/B \simeq \mathbb{P}^1$

SL_2-equiv on $\mathbb{P}^1 \longleftrightarrow \mathbb{Z}$.

$\mathcal{O}(n) \longleftrightarrow n$.

(can ~~the~~ construct as by its sections: homogeneous poly in $[x:y]$ $\rightarrow$ automatically has $SL_2(\mathbb{C})$ action).

Thm (Borel-Weil) Let $\lambda \in P$, Then

$$\Gamma(G/B, \mathcal{L}_\lambda) = \begin{cases} L(\lambda), & \lambda \in P_+ \\ 0 & \text{otherwise}. \end{cases}$$

Pf: Let $V = \Gamma(G/B, \mathcal{L}_\lambda)$, is f.d. by general result of AG.

~~Strategy: find highest weight vector: killed by $\mathfrak{n}_+$, i.e. fixed by N.~~

Consider space $V^{\mathfrak{n}_+} = V^N = \{ v \in V \mid gv = v \ \forall g \in N \}$.

For $X = G/B$, have open cell $X_{w_0} = B\dot{w}_0 B/B = N\dot{w}_0 B/B \simeq N$, which N acts freely, transitively on.

have restr map $\Gamma(X, \mathcal{L}_\lambda) \hookrightarrow \Gamma(X_{w_0}, \mathcal{L}_\lambda) =: \tilde{V}$

We claim ~~that~~ $\tilde{V}^N$ is 1-dim w/ weight λ.

clear: N acts freely, transitively on X_{w_0}, so section is determined by value at a pt. : $\Gamma(X_{w_0}, \mathcal{L}_\lambda)^N \simeq$ fiber of $\mathcal{L}_\lambda$ at x_0.

choose $x_0 = \dot{w}_0 B$ ...

T acts on fiber by $w_0(\lambda)$ ...?

[will clarify next time]

note: id. B is the smallest cell.

Thus, there are ~~2 possibl~~ two possibilities

(1) $V^N \simeq \mathbb{C}_\lambda$ is 1-dim, so $V \simeq L(\lambda)$.

(2) $V^N = 0$, so $V \simeq 0$.

<u>Lemma</u>: (1) happens iff $\lambda \in P_+$.

If $\lambda \notin P_+$, then $L(\lambda)$ is ∞-dim, so $L(\lambda) \neq V$.

converse, note $X - X_{w_0} = \bigcup D_i$, D_i closed irred, codim $= 1$.

A section s of line bundle $\mathcal{L}$ on X_{w_0} extends to x_0

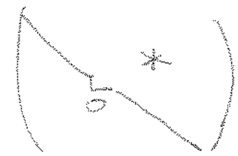
$\iff$ s has no poles on D_i.

If have $P_i \subset X$, $\dim P_i = 1$, s.t. $[P_i]\cdot[D_i] > 0$.

then s has no poles on D_i $\iff$ s is regular on P_i $\forall P_i$ ~~$\forall$ such P_i~~.

just on P_i is enough, since a mero fn on X would have poles along codim 1 thing. So if doesn't have pole ~~exce~~ at some pt in D_i, Then it doesn't have pole along D_i at all

For G/B, $D_i = B s_i w_0 B/B$, $\mathbb{P}_i = P_i/B \simeq \mathbb{P}^1$ (for $\cancel{SL_n(\mathbb{C})}$ $SL(n)$, P_i.



($\mathrm{Lie}(P_i) = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_i}$.

$$\mathcal{L}_{\mathbb{R}} \mathcal{L}_\lambda\big|_{\mathbb{P}_i} \simeq \mathcal{O}(\underline{\langle \lambda, h_i \rangle}).$$

≥ 0 by hypothesis,
and since s is nonzero
outside D_i, it must has
zero of that order.