Lie 31/10/17.

Recall: Lie alg.

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

$R \subset \mathfrak{h}^*$

$W = \langle s_i \rangle$

Alg. ~~Lie~~ gp

G = semisimple

T = max torus, $Lie(T) = \mathfrak{h}$

B = max conn. solv. subgp.

$Lie(B) = \mathfrak{b}$

$N(B) = B$, $N(T)/T = W$, generally no ~~emb.~~ lift. $W \hookrightarrow N(T) \xrightarrow{} W$, id

G/B is proj (thus complete)., 'flag variety'.

For $\mathfrak{g} = sl_n$, $X \simeq$ set of flags in $\mathbb{C}^n$.

$\mathfrak{g} = sl_2$, $X \simeq \mathbb{P}^1$.

Cell decomp of G/B.

For $\mathfrak{g} = sl_2$, $\mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}$. $\Rightarrow H^*(\mathbb{P}^1)$, $H^i_{BM,*} = \begin{cases} \mathbb{C} & i=0 \\ 0 & \text{else} \\ \mathbb{C} & i=2 \end{cases}$.

"$\partial \mathbb{C} = \varphi$" in H_*".

Some facts abt alg. groups.

(1) For any root $\alpha \in R$, $\exists$ subgp $G_\alpha \subset G$, $Lie(G_\alpha) = \langle e_\alpha, f_\alpha, h_\alpha \rangle$.

"$(sl_2)_\alpha$ integrates to a closed subgp".

(Pf: $G_\alpha = Stab(\alpha^\perp) = \{ g \in G \mid Ad_g . h = h \ \forall h \in \alpha^\perp \}$.

$\{h \in \mathfrak{h} \mid \langle \alpha, h \rangle = 0\}$.)

Moreover, $G_\alpha \simeq SL_2 / PGL_2$.

In particular, we have $\forall \alpha \in R$, a hom.

$u_\alpha : \mathbb{C} \to G_\alpha \hookrightarrow G$.

$t \xmapsto{\exp} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (note "exp is alg. because the expansion terminates)

(2) Let N = unipotent radical of B

$Lie(N) = \mathfrak{n}_+$.

Thm: For any choice of ordering of +ve roots. $R_+ = \{\beta_1, \dots, \beta_m\}$, $m = |R_+|$.

~~consider~~ morphism $\mathbb{C}^m \to N$

Then $(t_1, \dots, t_m) \mapsto u_{\beta_1}(t_1) \cdot \dots \cdot u_{\beta_m}(t_m)$ is isom of alg. varieties.

Example $G = SL_3$. $N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ =

$$\begin{pmatrix} 1 & t_1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ & 1 & t_2 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & t_3 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Pf idea: $\mathfrak{n} \underset{\log}{\overset{\exp}{\rightleftarrows}} N$. usually these are defined in some open nbhd of 0/1. but since $\mathfrak{n}$ is nilpotent, and N unipotent, these are algebraic maps, defined on whole $\mathfrak{n}$, N, so it gives isom.

Now, let us study G/B.

More precisely, consider the double cosets $B\backslash G/B$ = $B\times B$ orbits ~~of~~ in G.

Thm: (1) $B\backslash G/B = W$, namely each $B\times B$ orbit in G is

$C(w) = B\dot{w}B$, where $\dot{w} \in N(T)$ is rep of $w \in N(T)/T = W$.

$$G = \coprod_{w\in W} B\dot{w}B$$

(2) $G/B \simeq \coprod C'(w)$, $C'(w)$: image of $B\dot{w}B$ in G/B, each $C'(w) \simeq \mathbb{C}^{l(w)}$ } Bruhat decomp.

Example $G = SL_2$.

Then $G = B \sqcup B\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}B$

$= B_* \sqcup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_- B_+$

$A = L\cdot U$ iff $A \neq \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. "Gaussian elimination".

$= \cancel{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sqcup}\ B_-B_+ \sqcup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_+$.

In G/B, B_-B_+ gives $\mathbb{C}$ — comes from $w=-1$, $l(w)=1$

$\mathbb{P}^1 = \mathbb{C} \cup pt$.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}B_+$ gives pt. — $w = id$, $l(w)=0$.

Pf: see "Springer".

want to know how they're parametrize by roots ..., i.e. how the roots come into play. where $l(w)$ come from?

Why $\mathbb{C}^{\ell(w)}$?

For $w \in W$, get $R_+ = R_w \sqcup R_w'$, $|R_w| = \ell(w)$.

R_w: $w(\alpha) \in R_-$, R_w': $w(\alpha) \in R_+$.

Let us consider $C(w) = B\dot{w}B = N\dot{w}B$. $(B = NT)$.

so $C(w) = \{ u_{\beta_1}(t_1) \cdots u_{\beta_m}(t_m) \cdot \dot{w} \cdot B \}$

Choose the ordering so that $\beta_1 \cdots \beta_\ell \in R_w$, $\beta_{\ell+1} \cdots \beta_m \in R_w'$

$= \{ u_{\beta_1}(t_1) \cdots u_{\beta_\ell}(t_\ell) \cdot u_{\beta_{\ell+1}}(t_{\ell+1}) \cdots u_{\beta_m}(t_m)\, \dot{w} \cdot B \}$.

$= \{ u_{\beta_1}(t_1) \cdots u_{\beta_\ell}(t_\ell) \cdot \dot{w}\, \underbrace{u_{w(\beta_{\ell+1})}(t_{\ell+1}) \cdots u_{w(\beta_m)}(t_m)}_{B} \cdot B \}$.

$w(\beta_i)$

$= \{ u_{\beta_1}(t_1) \cdots u_{\beta_\ell}(t_\ell)\, \dot{w}\, B \}$.

(technically $w u_\alpha(t) w^{-1} = u_{w(\alpha)}(\pm t)$ some k. $w G_\alpha w^{-1} = G_{w(\alpha)}$.)

(2) is known as "Bruhat decomp".

Each of $C(w) \cong \mathbb{C}^{\ell(w)}$ is smooth, locally closed.

Q: What is closure?

Since these ~~things are~~ orbits, ~~they~~ their closure is union of orbits.

$$\overline{C(w)} = C(w) \sqcup \coprod_i C(w_i)$$

Thm: $C(w_1) \subset \overline{C(w_2)}$ iff $w_1 \leq w_2$ in Bruhat order.

"algebra & geometry agree".

among the many defs, the one useful here is "take w_2 = reduced $w_2 = s_{i_1} \cdots s_{i_k}$, reduced, w_1 can be obtained by deleting terms.

Warning: Closures of cells can be non-smooth. (but is always closed subvariety).

"Schubert cells".