Lie 25/10/17.

$\mathfrak{g} \longrightarrow G$

algebra $\longrightarrow$ (algebraic) geometry.

$\mathfrak{g}$ s.s./$\mathbb{C}$

$\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+$. $\longrightarrow$?

$R \subset \mathfrak{h}^*$, $\alpha \in R$

$P_{\mathbb{Z}} \subset \mathfrak{h}^*$ weight lattice $\quad h_\alpha \in \mathfrak{h} \quad W : \mathfrak{h}^* \circlearrowleft$

G

Algebraic groups /$\mathbb{C}$.

(for $\mathfrak{g}$ s.s., Lie gps $\leftrightarrow$ Alg gp.)

- Linear alg. gp. (closed) subgp in $GL(V)$. (in Zariski).
- Repn : $G \longrightarrow GL(\ldots)$ polynomial. aka "rational repn"
- we have notion of semi simple elmt in G, unipotent, radical.

(any $g \in G$ can be written as $g_s g_u = g_u g_s$ (analog of Jordan decomp).

- torus $T \subset G$ is subgp ~~≅~~ isom to $(\mathbb{C}^*)^r$.

can define $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ $(\simeq \mathbb{Z}^r)$.

maximal torus

- Borel subgp : $B \subset G$, conn. & solvable & maximal

e.g. $G = GL(n) / SL(n)$, then B = upper tri.

(note: Borel subgp came before Borel subalg.

[Ref: Linear Alg. Gps. (Springer)

Properties ~~of~~ of B (Fixes $B \subset G$ Borel.)

① G/B is projective (thus complete).

moreover, G/P ~~is~~ complete $\iff P \supset gBg^{-1}$.

② Any two Borel subgps are conjugate.

③ $N_G(B) = B$.

normalizer $= \{g \mid gBg^{-1} = B\}$.

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complete is analog of compact in AG.

e.g. quasi-proj var is complete iff it is proj.

open $\subset$ proj var.

global ~~fu~~ function on complete are const.

~~Example: G =~~

Cor: ②+③ $\{\text{Borel subgp} \subset G\} \longleftrightarrow G/B$

Example: $G = GL(n)$, $B = \begin{pmatrix} & \\ 0 & \end{pmatrix}$ (upper triangular).

G/B = flag variety $= \{F = \{F_0, \dots, F_n\} \mid 0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n\}$, $\dim F_i = i$.

'standard flg' $F^\circ = \{0 \subset \langle e_1 \rangle \subset \dots \subset \langle e_1, \dots, e_n \rangle\}$.

$\mathrm{Stab}_G(F^\circ) = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & & * \end{pmatrix} \right\} = B$ + G acts ~~transitively~~ transitively on flag variety.

e.g. $n=2$, $G/B = \mathbb{P}^1$.

Hence G/B is sometimes ~~used~~ still called flag var. even for other G's.

Properties of max tori:

(G s.s).

(1) Every maximal torus $\subset$ some Borel

Every Borel $\supset$ ~~some~~ at least 1 max torus.

(2) All max tori are conjugate.

- e.g. $C_G(T) = T$

  " centralizer

- $N_G(T)/T = W$ is finite

- If $B \supset T$, then we can write $B = T \cdot N$, where

  $T \times N \longrightarrow B$ is isom of alg. var.

  N = unipotent radical of B.

  (e.g. $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, T = diag., $N = \begin{pmatrix} 1 & * \\ & \ddots \\ 0 & & 1 \end{pmatrix}$).

<u>Cor</u>: $\{\text{pairs } (B \supset T)\}$ ~~$\{B \supset T\}$~~

$$\begin{array}{c} \{\text{pairs } (B \supset T)\} \\ \downarrow \pi \\ \{T\} \end{array}$$

$\pi^{-1}(T) \longleftrightarrow W$. [exercise: check this]

philosophically: to define "diagonal matrices" need a basis.

to define "upper triangular + diag" need an ordered basis.

W is "ways to choose ordering".

<u>Example</u>: $G = GL(n)$, T = diag.

$G = SL(n)$

subtlety: see $SL(2)$, say $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$g \begin{pmatrix} a & \\ & b \end{pmatrix} g^{-1} = \begin{pmatrix} b & \\ & a \end{pmatrix}$.

so g acts by (12).

but ~~W~~ $g^2 \neq 1$,

so ~~$W \hookrightarrow$~~ W doesn't lift into subgp of $N_G(T)$.

$$\begin{array}{c} N_G(T) \\ \downarrow \uparrow \not\exists \text{ (of subgp)} \\ W \end{array}$$

Thm: Let G be ~~complete~~ connected s.s. linear alg. gp.

Then (1) $\mathfrak{g} = \mathrm{Lie}(G)$ is s.s. Lie alg.

(2) If $T \subset G$ is max torus,

then $\mathrm{Lie}(T) = \mathfrak{h}$ Cartan.

and any Cartan is obtain this way.

(3) If $B \supset T$, then $\mathrm{Lie}(B) = \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ for some choice of polarization.

(4) $N_G(T)/T$ = Weyl gp.

Conversely, given $\mathfrak{h} = \mathrm{Lie}(T)$ + choice of polarization, there is corresponding Borel

<u>Thm</u> Given s.s. Lie alg. $\mathfrak{g}$, $\exists$ connected linear alg. gp G w/ $\mathrm{Lie}(G) = \mathfrak{g}$.

G is ss..

If require that G is s. conn, then G is unique.

not very natural in AG...

More generally: conn. gps G w/ $\mathrm{Lie}(G) = \mathfrak{g}$

$\updownarrow$

$X(G) = X(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$

"root datum" $\left\{ \begin{array}{l} X(T) \text{ is lattice in } \cancel{\mathfrak{t}^*}\ \mathfrak{h}^* \\ \text{and } \underset{\text{adj.}}{Q} \subseteq X \subseteq \underset{\text{s.c.}}{P} . \end{array} \right.$

eg. $\mathfrak{g} = \mathfrak{sl}_2$.

$\underset{PGL_2}{\underset{2\mathbb{Z}}{\overset{Q}{\Vert}}} \subset \quad \subset \underset{SL_2}{\underset{\mathbb{Z}}{\overset{P}{\Vert}}}$

[usually we will care abt the s.c. one].