Lie 24/10/17

~~R~~ Recall BGG-resolution

$\lambda \in P_+$, $L(\lambda)$ - f.d.

$$\longrightarrow \bigoplus_{\ell(w)=k} M(w.\lambda) \longrightarrow \dots \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0 .$$

Have shown $\cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow \mathbb{C}$. 'standard resolution', used to compute cohom. of Lie alg.

$$D_i = U\mathfrak{g} \otimes_{U\mathfrak{b}} \Lambda^i(\mathfrak{g}/\mathfrak{b}) \underset{v.s}{\cong} U\mathfrak{n}_- \otimes_{\mathbb{C}} \Lambda^i \mathfrak{n}_-$$

projection over $\mathfrak{n}_-$ (of $\mathfrak{n}_-$).

Claim 1 : D_i has composition series w/ factors $M(\lambda_i)$, λ_i weights of $\Lambda^i(\mathfrak{g}/\mathfrak{b})$.

Pf: in words, $\mathfrak{b}$ is solvable, so its irreps are 1-dim, so has comp. series, factors are 1-dim. the factors give $M(\lambda_i)$.

Now project $D_i \longrightarrow \mathbb{C}$ onto $\mathcal{O}_0$.

$$\longrightarrow (D_1)^0 \longrightarrow (D_0)^0 \longrightarrow \mathbb{C} \longrightarrow 0$$

$\dots \quad \overset{\shortparallel}{C_1} \quad \overset{\shortparallel}{C_0}$

. $C_i := (D_i)^0$ is the proj of D_i onto $\mathcal{O}_0$.

(i.e. the $\mathcal{O}_0$ component).

(projection is exact).

why? D_i's have a lot of Verma modules which are irrelevant.

Now each C_i will have comp. series, where factors are $M(\lambda)$, λ weight of $\Lambda^i \mathfrak{n}_-$, & $\lambda \in W.0$.

problem has reduced to "combinatorics": counted weights in $\Lambda^i \mathfrak{n}_-$.

$\{$weights of $\Lambda^i \mathfrak{n}_-\}$

$\shortparallel$ basis f_α, $\alpha \in R_+$

$\{-\sum_S \alpha \mid S \subset R_+, |S| = i\}$

$w.0 = -\sum_{\alpha \in R_+,\ w^{-1}(\alpha) \in R_-} \alpha$

$\left(= \frac{1}{2} w(\sum \alpha) - \frac{1}{2}(\sum \alpha) \right.$

$\left. = \frac{1}{2}\left(\sum_{w^{-1}(\alpha) \in R_+} \alpha - \sum_{\alpha \in R_+} \alpha \right) \right).$

eg $\mathfrak{g} = \mathfrak{sl}_3$.

e.g. $\mathfrak{g} = \mathfrak{sl}_3$.

$i=0$	0	0
$i=1$	$\{\alpha_1\}$	α_1
	$\{\alpha_2\}$	α_2
	$\{\alpha_1+\alpha_2\}$	$\alpha_1+\alpha_2$
$i=2$	$\{\alpha_1, \alpha_2\}$	$\alpha_1+\alpha_2$
	$\{\alpha_1, \alpha_1+\alpha_2\}$	$2\alpha_1+\alpha_2$
	$\{\alpha_2, \alpha_1+\alpha_2\}$	$\alpha_1+2\alpha_2$
$i=3$	$\{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$	$2\alpha_1+2\alpha_2$.

w	~~w.0~~	$w.0$
1	0	0
s_1	α_1	α_1
s_2	α_2	α_2
s_1s_2	$\alpha_1, \alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$
s_2s_1	$\alpha_2, \alpha_1+\alpha_2$	$\alpha_1+2\alpha_2$
$s_1s_2s_1$	$\alpha_1, \alpha_2, \alpha_1+\alpha_2$.	$2\alpha_1+2\alpha_2$.

<u>Lemma</u> $\{\text{wghts of } \Lambda^i \mathfrak{n}_- \text{ in } W.0\} = \{w.0 \mid \ell(w) = i\}$.

(~~each wght of~~ even as a set w/ multiplicity

<u>Pf</u>: Left to reader.

~~Thus, get we~~

Thus, we get resolution $\cdots C_2 \to C_1 \to C_0 \to \mathbb{C} \to 0$

C_k has composition series w/ factors $M(w.0)$, $\ell(w)=k$, each appearing once.

<u>Lemma</u>: $(\lambda \in P_+)$ $\mathrm{Ext}^1(M(w_1.\lambda), M(w_2.\lambda)) = 0$, unless $w_1 \geq w_2$ in Bruhat order.

Cor: $C_k = \bigoplus_{\ell(w)=k} M(w.0)$.

So we are done.

Then apply translation functors to get statement for others $\lambda \in P_+$.

Cor.: $H^k(\mathfrak{n}_-, \mathbb{C}) = \mathrm{Ext}^k_{\mathfrak{n}_-}(\mathbb{C}, \mathbb{C})$.

(Note $C_\bullet \to \mathbb{C}$ is a $\mathfrak{n}_-$-free res. of $\mathbb{C}$).

$= H^k(\mathrm{Hom}_{\mathfrak{n}_-}(C_\bullet, \mathbb{C}))$.

$= H^k(\cdots \xrightarrow{0} (\bigoplus_{\ell(w)=k} \mathbb{C}) \xrightarrow{0} \cdots)$

$= \bigoplus_{\ell(w)=k} \mathbb{C} = \mathbb{C}^{|\{w \mid \ell(w)=k\}|}$.

$\mathrm{Hom}_{\mathfrak{n}_-}(M(\mu), \mathbb{C} V)$

$= \cancel{\mathrm{Hom}_{\mathfrak{n}_-}(V_\mu, V)}$

$= V$ ($M_{\mathfrak{g}}(\mu)$ is free/$\mathfrak{n}_-$, v_μ is highest wght, ~~and~~ since we only care abt $\mathfrak{n}_-$ (not $\mathfrak{b}$), so wghts spaces don't have to be preserved. so $v_\mu \otimes$ can be sent to anything).

(first computed by Bott, ~60 years ago, using geometry, diff forms on some var.).

can think of $D_\bullet \to \mathbb{C}$ as resolving ~~different~~ constant sheaf by differential form.