李 19/10/17

Recall : (1) $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/W_\cdot} \mathcal{O}_\lambda$ simple obj $L(w\cdot\lambda) \in \mathcal{O}_\lambda$

Verma $M(w\cdot\lambda) \in \mathcal{O}_\lambda$.

(2) if $\lambda, \mu \in P_+$, then $\mathcal{O}_\lambda \simeq \mathcal{O}_\mu \simeq \underset{\mathbb{C}}{\underset{\cup}{\mathcal{O}_0}}$.

Today : BGG resolution & Weyl Character formula.

↳ compute $c_{w'w}$ when $\lambda \in P_+$.

Thm $\lambda \in P_+$. Then $\exists$ resolution (BGG resolution).

$$0 \longrightarrow C^N \longrightarrow C^{N-1} \longrightarrow \cdots \longrightarrow C^1 \longrightarrow C^0 \longrightarrow L(\lambda) \longrightarrow 0$$

where $C^k = \bigoplus_{\substack{w \in W \\ \ell(w)=k}} M(w\cdot\lambda)$.

(In part., $C^0 = M(\lambda)$, $C^1 = \bigoplus_i M(s_i\cdot\lambda)$, so

$C^1 \longrightarrow C^0 \longrightarrow L(\lambda) \longrightarrow 0$ we've seen before).

Before proof, some eg./cor :

Eg: (1) $\mathfrak{g} = \cancel{sl_2}\ sl_2$

$$0 \longrightarrow M(-\lambda-2) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0 .$$

(2) $\mathfrak{g} = sl_3$.

$$\begin{tikzcd}
 & s_1\cdot\lambda \arrow[r] & \lambda & \\
s_1s_2\cdot\lambda \arrow[ur] \arrow[rrr] & & & s_2\cdot\lambda \arrow[ul] \\
 & w_0\cdot\lambda \arrow[ul] \arrow[r] & s_2s_1\cdot\lambda \arrow[ur] \arrow[uul] &
\end{tikzcd}$$

$$0 \longrightarrow \underset{\ell(w)=3}{M(w_0\cdot\lambda)} \longrightarrow \underset{\ell(w)=2}{M(s_1s_2\cdot\lambda) \oplus M(s_2s_1\cdot\lambda)} \longrightarrow \underset{\ell(w)=1}{M(s_1\cdot\lambda) \oplus M(s_2\cdot\lambda)} \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

Cor: (Weyl character formula) ($\lambda \in P_+$).

recall
$ch(M) = \sum (\dim M_\lambda)\cdot e^\lambda$
$e^\lambda \in \mathbb{C}[P]$.

$$ch(L(\lambda)) = \sum_{w \in W} (-1)^{\ell(w)} ch(M(w.\lambda))$$
$$= \sum (-1)^{\ell(w)} e^{w.\lambda} \cdot ch(M(0))$$
$$= \sum \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R^+} (1-e^{-\alpha})}$$

follows from PBW

Formula has many useful corollaries.

In part., $\lambda = 0$, we get

$$1 = \underbrace{\frac{\sum (-1)^{\ell(w)} e^{w(\rho)}}{\prod e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}}}_{\text{Weyl denominator identity}}$$

"Weyl denominator".

$$= \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha \in R^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})}.$$

$$= \frac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum (-1)^{\ell(w)} e^{w(\rho)}}.$$

Remark: Weyl pap. (first?) proved before BGG.

e.g. $\mathfrak{g} = \mathfrak{sl}_2$ $(z := e \ldots)$

$$ch(L(n)) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} = z^n + z^{n-2} + \ldots + z^{-n}.$$

e.g. $\mathfrak{g} = \mathfrak{sl}_n$. Denominator identity is

$$\prod_{i>j}\left(\frac{x_i}{x_j} - \frac{x_j}{x_i}\right) = \sum_{s \in S_n} (-1)^{sgn(s)} (\ldots)$$

use Vandermonde determinant formula

$$\det\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots \\ 1 & t_2 & t_2^2 & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

} exercise: figure out the ~~details~~ details

Warning: w/ nice formula (of BGG), might be ~~tempted~~ to expect when $\lambda \notin P_+$, ~~it is so that~~ there will be some nice resolution.
False: There are λ s.t. there ~~are~~ is no resolution by Verma module ...

Before proving BGG, need some reminders on homological algebra.

in any abelian cat $\mathcal{C}$, $\mathrm{Hom}(X,\cdot)$ is left exact, so have $R^i(\mathrm{Hom}(X,\cdot)) =: \mathrm{Ext}^i(X,\cdot)$

$\mathrm{Hom}(\cdot, Y)$ is right exact, $L^i(\mathrm{Hom}(\cdot,Y)) = \mathrm{Ext}^i(\cdot, Y)$.

To compute $\mathrm{Ext}^i(X,Y)$, replace X by a ~~projective~~ resolution

$$\underbrace{\cdots \to P_1 \to P_0}_{P_\bullet} \to X \to 0$$

so $\mathrm{Ext}^i(X,Y) = H^i(\mathrm{Hom}(P_\bullet, Y))$

works whenever have enough projectives (injectives),

~~but~~ sometimes cat of finite ~~dim~~ dim stuff may not have enough projectives...

'special case': Cat. of reps of Lie alg. $\mathfrak{a}$ $\quad$ a a a $\mathfrak{a}$ a aa $\mathfrak{a}$

$$\mathrm{Ext}^i_{\mathfrak{a}}(\mathbb{C}, L) =: H^i(\mathfrak{a}, L) \quad (\text{cohom of } \mathfrak{a} \text{ w/ coeff in } L).$$

(e.g. $H^0(\mathfrak{a}, L) = \mathrm{Hom}_{\mathfrak{a}}(\mathbb{C}, L) = L^{\mathfrak{a}}$ = vectors killed by $\mathfrak{a}$.

Note: if $\mathfrak{a}$ s.s., not much to study...

e.g. $\mathrm{Ext}^1_{\mathfrak{a}}(\mathbb{C}, L) = 0$ for L = f.d. (no non-trivial extension).

in fact, all higher Ext vanish.

projective is something "almost free",

& in our case, an example ~~a~~ would be free mod of over ~~$U\mathfrak{a}$~~ $U\mathfrak{a}$. ?.

Claim: There is a standard projective resolution of $\mathbb{C}$.

$$P_k = U\mathfrak{a} \otimes_{\mathbb{C}} \Lambda^k \mathfrak{a} \qquad \mathfrak{a} \subset U\mathfrak{a} \otimes \Lambda^k \mathfrak{a}, \qquad \begin{matrix} \mathfrak{a} \subset U\mathfrak{a} \text{ left mult;} \\ \mathfrak{a} \subset \Lambda^k \mathfrak{a} \ \cancel{\ } \ 0. \end{matrix}$$

$$\partial(u \otimes x_1 \wedge \cdots \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k$$
$$+ \sum (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k$$

$$\begin{array}{ccccccc} P_2 & & P_1 & & P_0 & & \\ U\mathfrak{a}\otimes \mathfrak{a}\wedge\mathfrak{a} & \longrightarrow & U\mathfrak{a}\otimes_{\mathbb{C}}\mathfrak{a} & \longrightarrow & U\mathfrak{a} & \longrightarrow U\mathfrak{a}/\mathfrak{a}U\mathfrak{a} \cong \mathbb{C} & \to 0 \\ & & u\otimes x & \longmapsto & ux & & \end{array}$$

$u\otimes x_1\wedge x_2 \longmapsto ux_1\otimes x_2 - ux_2\otimes x_1$ $\;u\otimes[x_1,x_2]$

exercise: check ~~that~~ this is resolution.

use PBW, i.e. pass to graded alg. etc.

~~@~~ *example*: $\mathfrak{a}$ abelian

$$P_k = S\mathfrak{a}\otimes\Lambda^k\mathfrak{a} \quad ..$$

[ka-shoo]

Claim Then $P_\bullet$ is dual of Koszul complex

"

diff forms on v. space V

w/ poly coeff.

[exercise].

$$S\mathfrak{a}^* \otimes \Lambda^k\mathfrak{a}^*$$

poly. $\quad$ diff.

Want to use $P_\bullet \to \mathbb{C}$

to get BGG,

but first need to modify $P_\bullet \to \mathbb{C}$.

Recall $\mathfrak{g} = \mathfrak{n}_-\oplus\mathfrak{b}$. (~~works~~ works whenever $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{b}$, $\mathfrak{a},\mathfrak{b}$ are subalg. - $\oplus$ as v.s.).

new cx: $P_k = U\mathfrak{g}\otimes_{U\mathfrak{b}}\Lambda^k(\mathfrak{g}/\mathfrak{b})$. (as v.s. $\simeq U\mathfrak{n}_-\otimes\Lambda^k\mathfrak{n}_-$).

~~Claim~~ $\partial: P_k \to P_{k-1}$ same formula.

$P_\bullet \to \mathbb{C}$ $\qquad$ "relative ~~Lie~~ cohomology"

$\mathfrak{g}$-mod.,

free over $\mathfrak{n}_-$ (not projective over $\mathfrak{g}$-mods...)

Now $P_\bullet \in \mathcal{O}$; in fact P_k has finite comp. series

whose factors are Verma mod. $M(\lambda)$,

$\lambda \in$ wght$(\Lambda^k(\mathfrak{g}/\mathfrak{b}))$.

More generally,

if $M = U\mathfrak{g} \otimes_{U\mathfrak{b}} L$, L a f.d. $\mathfrak{b}$-mod.,

then M has comp series, w/ factors $M(\lambda)$,

$\lambda \in$ weight(L).

~~R~~ Rk Weyl gp doesn't appear in $P_\bullet$ yet . . .