李 17/Oct 10/17

Recall: $\mathcal{O}$-category "gen by h.w. mod"

$\mathcal{O} = \bigoplus_{\lambda\in\mathfrak{h}^*/W.} \mathcal{O}_\lambda$, $\mathcal{O}_\lambda = \{ M \mid (z-\chi_\lambda(z)) \text{ is nilpotent on } M,\ z\in Z(U\mathfrak{g}) \}$.

χ of $M(\lambda)$.

has simple obj. $L(w.\lambda)$, $w\in W$.

also contains $M(w.\lambda)$, $w\in W$.

$$[M(w.\lambda)] = [L(w.\lambda)] + \sum_{w'>w} c_{ww'} [L(w'.\lambda)]$$

(if $\lambda\in P_+$). Bruhat

Main goal today: prove

Thm For any $\lambda,\mu\in P_+$, we have equivalence of (abelian) cat.

$$\mathcal{O}_\lambda \simeq \mathcal{O}_\mu .$$

[In part., $\simeq \mathcal{O}_0$ — 'the regular block']

Pf: Use 'Translation functors'.

Before that, ... Preliminaries:

① Consider L – f.d $\mathfrak{g}$-module.

Then $M \longmapsto L\otimes M$ is functor $\mathcal{O}\to\mathcal{O}$.

[Note: $\mathcal{O}$ is not tensor cat, $\otimes$ violates fin generation: (look at character)]

Moreover, it is exact (just see on level of v.s.)

Properties: ① $\mathrm{Hom}_{\mathfrak{g}}(L\otimes A, B) \simeq \mathrm{Hom}_{\mathfrak{g}}(A, L^*\otimes B)$ usual dual, not $L^\vee$.

② $L\otimes M(\lambda)$ has fin. comp. series w/ factors $M(\lambda+\nu)$, ν a wght of L, appearing $\dim L_\nu$ times.

eg. $\mathfrak{g} = \mathfrak{sl}_2$, $L=\mathbb{C}^2$, wghts ± 1.

$$\underset{M(\lambda+1)}{\mathbb{C}^2\otimes M(\lambda) \supset M(\lambda-1)} \supset 0$$

(λ generic, will have $\oplus$, eg. if $\lambda\notin$ integral)

[Exercise]

Pf: (1) obvious

(2) $M(\lambda) = Ug \otimes_{Ub} \mathbb{C}_\lambda$. $\quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_-$

$\mathbb{C}_\lambda : \; h|_{\mathbb{C}_\lambda} = \langle \lambda, h \rangle \, \mathrm{id}$

$\mathfrak{n}_+|_{\mathbb{C}_\lambda} = 0$.

$\cancel{Ug \otimes Ug \otimes_{\mathbb{C}} \mathbb{C}_\lambda}$

$\cancel{h\otimes v_\lambda - 1 \otimes \langle \lambda, h\rangle v_\lambda}, \; h \in \mathfrak{h}$.

$x \otimes v_\lambda, \; x \in \mathfrak{n}_+$

so $M(\lambda) \otimes_{\mathbb{C}} L = \cancel{Ug}\left(Ug \otimes_{Ub} \mathbb{C}_\lambda\right) \otimes_{\mathbb{C}} L$

$= Ug \otimes_{Ub} \left(\mathbb{C}_\lambda \otimes_{\mathbb{C}} L \right)$

Thus, suffices to check that $[\mathbb{C}_\lambda \otimes L] = \sum (\dim L_\nu) \cdot [\mathbb{C}_{\lambda+\nu}]$

as $\mathfrak{b}$-mod.

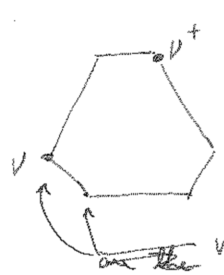
recall $\mathfrak{b}$ is solvable, so all irreps are 1-dim, in this case $\mathbb{C}_\mu$, $\mu \in \mathfrak{h}^*$.

(i.e. $\mathbb{C}_\lambda \otimes L$ has comp. series as $\mathfrak{b}$-mod w/ the right mult.

~~De~~ Def^n: Let $\lambda, \mu \in \mathfrak{h}^*$, $\nu = \mu - \lambda \in P$.

Let $L = L(\nu^+)$, where $\nu^+ \in W(\nu) \cap P_+$.

"irrep w/ extremal wght ν"



~~on the~~ $W\nu^+ = W\nu$ is the set of extremal wghts of L

~~Define~~ Define

$T_\mu^\lambda : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\mu$.

$M \longmapsto \pi_\mu(L \otimes M)$, $\quad \pi_\mu : \mathcal{O} \to \mathcal{O}_\mu$ is projection

T_μ^λ : BGG translation functors.

↳ will provide the desired equivalence.

Next

~~Properties~~

Properties of T_μ^λ:

(1) T_μ^λ is exact. (since $\mathcal{O} = \bigoplus \mathcal{O}_\lambda$, π_μ is exact too).

(2) Adjointness property: $\mathrm{Hom}(T_\mu^\lambda A, B) \simeq \mathrm{Hom}(A, T_\lambda^\mu B)$, $A \in \mathcal{O}_\lambda$, $B \in \mathcal{O}_\mu$.

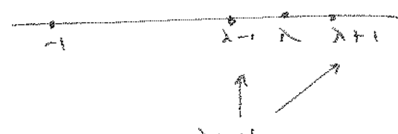
$$\mathrm{Hom}(L \otimes A, B) = \mathrm{Hom}(A, L^* \otimes B)$$

($\lambda - \mu$ is extremal wght of L^*).

Lemma Let $\lambda, \mu \in P_+$, $\nu = \lambda - \mu \in P$, $L = L(\nu^+)$.

Then for ν' a wght of L, $\lambda + \nu' \in W\cdot\mu \iff \nu' = \nu$,

e.g. $\mathfrak{g} = \mathfrak{sl}_2$, $L = \mathbb{C}^2$.



but they can't be in the same shifted W-orbit.

Pf: Exercise

Corollary: w/ assumption of Lemma,

$$T_\mu^\lambda M(\lambda) = M(\mu)$$

Pf: $L \otimes M(\lambda)$ has comp series w/ factors $M(\lambda + \nu')$

so $\pi\, T_\mu^\lambda M(\lambda) = \pi_\mu(L \otimes M(\lambda)) = \underbrace{M(\lambda+\nu)}_{\text{by Lemma}}$.

$T_\mu^\lambda M(w.\lambda) = M(w.\mu)$ ← apply analog of Lemma w/ $w.(P_+)$ instead of P_+.

$T_\lambda^\mu T_\mu^\lambda M(w.\lambda) = M(w.\lambda)$

$M = M_0 \supset M_1 \supset \dots \supset M_n = 0$

$X_i = M_i / M_{i+1}$, π exact

$\pi(M) = \pi(M_0) \supset \pi(M_1) \supset \dots \supset \pi(M_n) = 0$

$\underbrace{\pi(M_i)/\pi(M_{i+1}) = \pi(X_i)}_{\text{uses exactness}}$.

if π kills all factors X_i, but 1, then $\pi(M) = \pi(X_i)$

(if it didn't kill 2, then would still ruin semisimplicity).

Upshot: just study $\mathcal{O}_0$.

Next time: use geometry to get $c_{w'w}$