李 12/10/17.

Recall: $\mathcal{O}$: ① Abelian cat.

② Every obj has fin. length,
simple obj are $L(\lambda)$

③ has duality $M \mapsto M^\vee$.

Examples: $M(\lambda)$, $L(\mu)$, other hw modules.

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① Grothendieck group:

$K(\mathcal{O}) := \langle [M] \rangle_{M \in \mathcal{O}} \Big/$ if $0 \to A \to B \to C \to 0$ then $[A] - [B] + [C] = 0$

an abelian gp.

as far as dimension/character is concerned this is good.

"graded dimension".

Claim: $[L(\lambda)]$ form ~~basis~~ set of free generators in this gp.

$$\underset{K(\mathcal{O})}{[M]} = \sum [M : L(\lambda)] \underset{K(\mathcal{O})}{[L(\lambda)]}.$$

② Characters.

For $M \in \mathcal{O}$, define $ch(M) = \sum (\dim M_\lambda) \cdot e^\lambda \in \mathbb{C}[\mathfrak{h}^*]$.

(If all weights integral, can identify $\mathbb{C}[P] \simeq \mathbb{C}[z_1^{\pm 1}, \dots, z_r^{\pm 1}]$.

Note: if $0 \to A \to B \to C \to 0$, then $ch(B) = ch(A) + ch(C)$.

so ch is well def. on $K(\mathcal{O})$:

$$ch : K(\mathcal{O}) \longrightarrow \mathbb{C}[\mathfrak{h}^*].$$

Exercise: Prove $ch(L(\lambda))$ are l.i.

Hint: $ch(L(\lambda)) = e^\lambda + \sum_{\mu < \lambda} c_\mu e^\mu$

Thus, $ch(M) = \sum [M : L(\lambda)]\, ch(L(\lambda))$.

E.g., $ch(M(\lambda)) = \sum_{\underline{k}} e^{\lambda - k_\alpha \cdot \alpha} = e^{\lambda} \sum_{\underline{k}} e^{-k_\alpha \cdot \alpha} = e^{\lambda} \prod_{\alpha \in R_+} (1 + e^{-\alpha} + \dots) = \frac{e^{\lambda}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}$.

(By PBW, basis in $U\mathfrak{n}_-$ is $f_{\underline{k}} = \prod f_\alpha^{k_\alpha}$, $k_\alpha \in \mathbb{Z}_{\geq 0}$

$f_{\underline{k}} v_\lambda = \lambda - \sum k_\alpha \cdot \alpha$ $\qquad \underline{k} = (k_\alpha)_{\alpha \in R_+}$ ordered in some way.

③ ~~Blo~~ Blocks.

Let $\chi : Z(U\mathfrak{g}) \to \mathbb{C}$ central ~~ch~~ character.

Given $M \in \mathcal{O}$, let

$$\cancel{M^\chi}\ M^\chi = \{ v \in M \mid (z - \chi(z))^N v = 0 \ \ \forall z \in Z(U\mathfrak{g}) \text{ for } N \gg 0 \}.$$

$M = \bigoplus_\chi M^\chi$, each ~~$M^\chi$~~ M^χ is submodule.

["character": morphisms from $\mathfrak{g}$ ~~(?)~~ $\to$ scalars.]

E.g. $M = M(\lambda)$, then each z acts by scalar:

$$z|_{M(\lambda)} = \chi(z, \lambda) \cdot id.$$

(so each $\lambda \in \mathfrak{h}^*$ defines central char. $\chi_\lambda = \chi(\cdot, \lambda)$.

(In particular, ~~$M = M^{\chi_\lambda}$~~ $M(\lambda) = M(\lambda)^{\chi_\lambda}$.)

Def^n $\mathcal{O}_\chi := \{ M \in \mathcal{O} \mid M = M^\chi \}$ full subcategory

(Also use notation $\mathcal{O}_\lambda$ for $\mathcal{O}_{\chi_\lambda}$.

'blocks'

[cannot always ~~or~~ expect Z central elements to act semisimply.

so $(z - \chi(z))^N v = 0$ can't be helped]

Thm $\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^*/W\cdot} \mathcal{O}_\lambda$.

i.e. (1) each $M \in \mathcal{O}$ can be written as $M = \bigoplus M^{\chi}$.

(2) if $\chi \neq \chi'$, $M \in \mathcal{O}_\chi$, $M' \in \mathcal{O}_{\chi'}$.

then $\mathrm{Hom}_{\mathcal{O}}(M, M') = 0$ ← use central element

$\mathrm{Ext}^*_{\mathcal{O}}(M, M') = 0$ ← i.e. $0 \to M' \to ? \to M \to 0$

only $M' \oplus M$.

just split the middle thing, and look at χ

"no non-trivial interaction between different $\mathcal{O}_\chi$.

$0 \to M' \to X \to M \to 0$

$0 \to M' \to X^{\chi} \oplus X^{\chi'} \to M \to 0$.

So to understand $\mathcal{O}$, suffices to understand $\mathcal{O}_\chi$.

Block $\mathcal{O}_\lambda$: • simple obj.: $L(\mu)$, $\mu \in W\cdot\lambda$

$\leadsto$ $K(\mathcal{O}_\lambda) = \bigoplus_{\mu \in W\cdot\lambda} \mathbb{Z}\cdot[L(\mu)]$

• contains $M(\mu)$, $\mu \in W.\lambda$,

and $[M(\mu)] = [L(\mu)] + \sum_{\substack{\mu' \in W.\mu \\ \mu' < \mu}} c_{\mu\mu'}[L(\mu')]$ ← lower triangular w/ 1 on diag. in some ordering .

Thus , $[M(\mu)]$, $\mu \in W.\lambda$, are also free generators of $K(\mathcal{O}_\lambda)$.

$$[L(\mu)] = [M(\mu)] + \sum_{\substack{\mu' \in W\cdot\mu \\ \mu' < \mu}} d_{\mu\mu'}[M(\mu')] .$$

Eg. $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda \in \mathbb{Z}_{\geq 0}$ $L_1 = [L(\lambda)]$, $L_2 = [L(\lambda')]$ $\lambda' = -\lambda - 2$. Note: these multiplicities don't depend on λ .

$M_1 = [M(\lambda)]$, $M_2 = [M(\lambda')]$.

$L_1 = M_1 - M_2$, $L_2 = M_2$. $d: \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\begin{matrix} M_1 = L_1 + L_2 \\ M_2 = L_2 \end{matrix}$: $c: \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Let us consider $\lambda \in P_+$.

trivial stabilizer under W.

$\rightsquigarrow$ objects ~~$L(\lambda)$,~~ $L(w.\lambda) \longleftrightarrow W$.

and $[M(w.\lambda)] = \sum_{w' \geq w} c_{ww'} [L(w'.\lambda)]$ — how to compute ?

in Bruhat order.

(so $c_{ww'} = [M(w.\lambda) : L(w'.\lambda)]$.

Next week: explain why, for $\lambda \in P_+$, all $\mathcal{O}_\lambda$ blocks same structure .