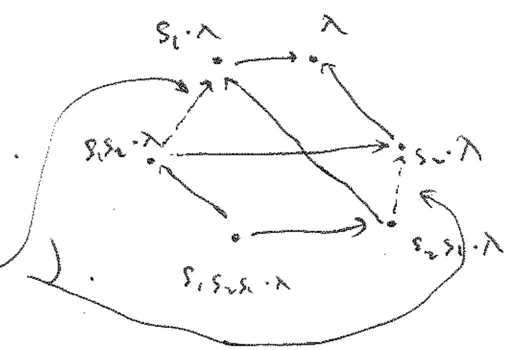
李 10/2/17.

Goal: understand str. of $M(\lambda)$.

recall
- any submod. $N \subset M(\lambda)$ ~~≠~~ contains maximal vec. & $M(\mu) \subset N \subset M(\lambda)$.
- $M(\mu) \hookrightarrow M(\lambda) \Rightarrow \mu \in W\cdot\lambda$, $\mu \leq \lambda$.
  (and $\dim \mathrm{Hom}(M(\mu), M(\lambda)) \leq 1$).
- Every $M(\lambda)$ has finite (length composition series),
  w/ ~~many~~ simple quotients $L(\mu)$, $\mu \in W\cdot\lambda$, $\mu \leq \lambda$.

Thm (Verma '66) If $\mu = s_\alpha \cdot \lambda$, $\langle \lambda+\rho, h_\alpha \rangle \in \mathbb{Z}_{>0}$,
then $M(\mu) \subset M(\lambda)$.

(Thm gives ).

Def: ① $\mu \uparrow_\alpha \lambda$ if $\alpha \in R_+$, $\mu = s_\alpha \cdot \lambda$, $\langle \lambda+\rho, h_\alpha \rangle \in \mathbb{Z}_{>0}$.

② μ is strongly linked to λ
if $\exists$ $\mu = \lambda_0 \uparrow_{\alpha_1} \lambda_1 \uparrow_{\alpha_2} \cdots \uparrow_{\alpha_n} \lambda_n = \lambda$

Note: in this case, $\mu \in W\cdot\lambda$, $\mu \leq \lambda$, but converse is <u>not true</u>.

Thm ① If μ is strongly linked to λ,
then $M(\mu) \subset M(\lambda)$, (thus $[M(\lambda) : L(\mu)] > 0$).

② (BGG, somewhere in a series of papers).

Conversely, if $[M(\lambda) : L(\mu)] > 0$, then μ is str. linked to λ,
so $M(\mu) \subset M(\lambda)$

NB:
Thm works for ~~non-dom integral~~ any weights.

Special case: $\lambda \in P_+$ ($\langle \lambda, h_\alpha \rangle \in \mathbb{Z}_{\geq 0}$?
$\langle \lambda+\rho, h_\alpha \rangle \in \mathbb{Z}_{>0}$. In particular, λ is regular).

Then $w_1 \cdot \lambda$ is str. linked to $w_2 \cdot \lambda$

iff $w_1 \geq w_2$ in Bruhat order on W.

→ says partial order of inclusion in $M(\mu) \subset M(\lambda)$ is the same as Bruhat order.

Pf: straightforward, use length of elem. of W etc.

Cor. $\lambda \in P_+$,
$$M(w_1 \cdot \lambda) \subset M(w_2 \cdot \lambda)$$
$$\Updownarrow$$
$$[M(w_2 \cdot \lambda) : L(w_1 \cdot \lambda)] > 0$$

Remark: as whether there is the inclusion doesn't dep. on λ. (as long it is in P_+)

Q: Are the numbers $[M(w_2 \cdot \lambda) : L(w_1 \cdot \lambda)]$ indep. of λ?

Category $\mathcal{O}$

name coined by BGG.

Def: Ob: modules over $\mathfrak{g}$, M

(1) M has wght dec.: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

(2) $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ acts locally finitely: for any $m \in M$, $\dim(U\mathfrak{b}) \cdot m < \infty$.

(3) M is f.g.

Eg. 1 $M(\lambda) \in \mathcal{O}$

Eg 2 Every h.w. mod $\in \mathcal{O}$.

Properties of $\mathcal{O}$.

(1) $\mathcal{O}$ is closed under taking quotients, $\oplus$, extensions

$0 \to A \to C \to B \to 0$, $A \in \mathcal{O}$, $B \in \mathcal{O}$ $\Rightarrow C \in \mathcal{O}$.

(2) Every $M \in \mathcal{O}$ is f. gen over $U\mathfrak{n}_-$

(PBW, $U\mathfrak{g} = U\mathfrak{n}_- \otimes U\mathfrak{b}$, apply $U\mathfrak{b}$ to gen + get f.d., then $U\mathfrak{n}_-$),

~~so $M = \oplus$~~

(3) Every $M \in \mathcal{O}$ has at least one maximal $v_\lambda \in M_\lambda$.

(4) Every $M \in \mathcal{O}$ has finite length, and simple mod. in $\mathcal{O}$ are $L(\lambda)$, $\lambda \in \mathfrak{h}^*$.

(not directly related to M is f.g.: e.g. $\mathbb{C}(x)$ as $\mathbb{C}[x]$-module).

Pf: induction on # gen. over $U\mathfrak{n}_-$ from finite length result abt $M(\lambda)$.

(cor of (4): $L(\lambda)$ generate $\mathcal{O}$).

(5) $\mathcal{O}$ is abelian cat.

What is structure of $\mathcal{O}$? Have the simple $L(\lambda)$....

more structure: Duality.

$$M^\vee = \oplus M_\lambda^*, \text{ w/ action}$$

$$\langle x\eta, m\rangle = \langle \eta, \tau(-x)m\rangle, \quad \eta \in M^\vee, \quad \tau : \mathfrak{g} \to \mathfrak{g}, \ \mathfrak{n}_\pm \to \mathfrak{n}_\mp, \ h \mapsto -h \text{ for } h \in \mathfrak{h} \quad \text{an involution}$$

eg. $\tau : e \mapsto f,\ f \mapsto e,\ h \mapsto h$ exercise(?): ~~construct~~ check for $\mathfrak{g}$. $e_i \mapsto f_i$, $f_i \mapsto e_i$

Exercise: ① Prove $L(\lambda)^\vee \cong L(\lambda)$

② $\cdot^\vee : \mathcal{O} \to \mathcal{O}$

③ $\cdot^\vee$ is exact contravariant functor.

④ ~~M is hw. mod. λ,~~

~~$\exists!\ M(\lambda) \twoheadrightarrow M$.~~

~~$\exists!\ M \hookrightarrow M(\lambda)^\vee$~~.

$\exists!\ M(\lambda) \twoheadrightarrow L(\lambda)$

$\exists!\ L(\lambda) \hookrightarrow M(\lambda)^\vee$.

"contragradient Verma mod"