李 5/19/17

Recall ① Shifted $W$-action

$$w.\lambda = w(\lambda+\rho)-\rho. \qquad \rho = \frac{1}{2}\sum_{\alpha\in R_+} \alpha.$$

Ref: Humphreys, ... Universal Env. Algebras.

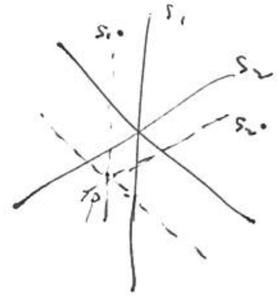
② $\lambda \in P_+ \Rightarrow M(\lambda) \supset M(s_i . \lambda)$.

Cor of HC:

$M(\lambda)$ has maximal weight/primitive ~~vec.~~ $v$ ($\mathfrak{n}_+ v = 0$) of weight $\mu$ for

$\Rightarrow$ $\mu \in W.\lambda$.

$\mu \le \lambda$. ($\lambda$ not nec. in $P_+$).



Pf. $(S\mathfrak{h})^W$ separate orbits:

if $\chi(\lambda) = \chi(\mu)$ $\forall \chi \in (S\mathfrak{h})^W$ then $\lambda \in W(\mu)$.

(use $Z(U\mathfrak{g}) \xrightarrow{\sim} (S\mathfrak{h})^W$, $z \longmapsto \chi_z$.

$z|_{M(\lambda)} = \chi_z(\lambda+\rho)$.

Cor 1 (of Cor).

① Any ~~Verma~~ h.w. module has fin length.

② Simple factors of ~~Verma~~ h.w. mod. $M(\lambda)$ are $L(\mu)$, $\mu \in W.\lambda$, $\mu \le \lambda$.

$V = V_0 \supset V_1 \supset \dots \supset V_n = 0$. $V_i/V_{i+1}$ = simple.

Pf: Induction on $d = \dim(M(\lambda)^{\mathfrak{n}_+}) < \infty$

$\cap$
$W.\lambda$

- $d=1$, easy.

- $d \ge 2$, take maximal vec. of ~~min pos~~ minimal weight. $L$.

$M/L$.

& Thus, can define multiplicities $[M(\lambda) : L(\lambda)] \in \mathbb{Z}_{\ge 0}$. "Jordan–Holder" thm

works in any situation where your objects have finite composition series.

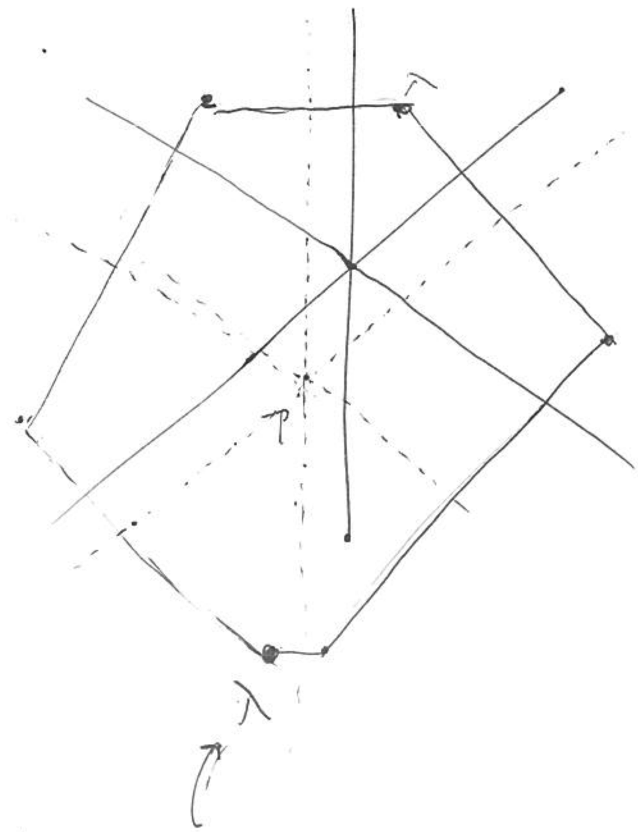
Question : how to compute mult. ?

$M(\mu) \hookrightarrow M(\lambda) \Rightarrow [M(\lambda) : L(\mu)] \geq 1$.

(~~note~~ remark : $\dim(\mathrm{Hom}(M(\mu), M(\lambda)) \leq 1)$ !).

i.e. if $M(\mu) \hookrightarrow M(\lambda)$ exists, it is unique ).

E.g. , . .



(1) . Let $\lambda$ st. $\lambda+\rho \in P_- = \{\lambda \in P \mid \langle\lambda, h_i\rangle \in \mathbb{Z}_{\leq 0}\} = w_0(P_+)$

Then $\lambda$ is minimal in $W_\bullet$ orbit. longest .

Pf : Induction ↓, (use the ~~more~~ defn of $l(w)$ as # simple roots sent to -ve .
to prove on $l(w)$ lemma : $l(w's_\alpha) > l(w') \Leftrightarrow w'\alpha > 0$
$w.\lambda \geq \lambda$
(easy for simple $\alpha = \alpha_i$) .

In fact : $M(\lambda)$ simple $\Leftrightarrow \forall \alpha \in R_+$, $\langle\lambda+\rho, h_\alpha\rangle \notin \mathbb{Z}_{>0}$ .

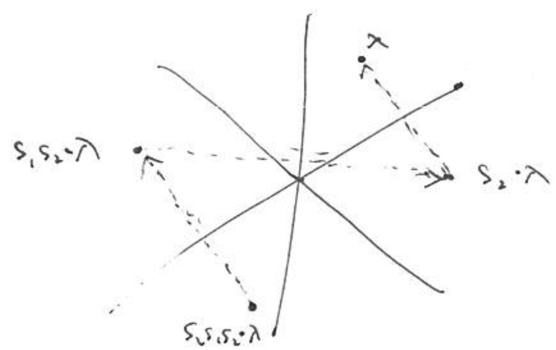
(2) $\lambda \in P_+$. $\quad w \in W$, $\quad M(w\cdot\lambda) \subset M(\lambda)$.

more precisely, $\quad w = s_{i_n} \cdots s_{i_1}$ reduced

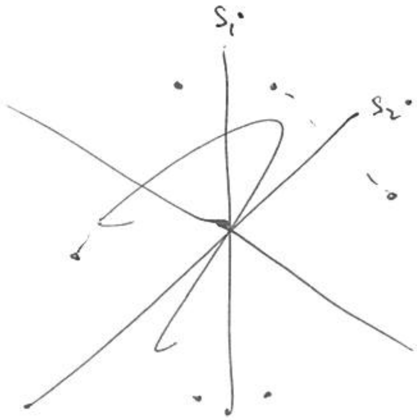
$$M(w\cdot\lambda) \subset M(s_{i_{n-1}}\cdots s_{i_1}\cdot\lambda) \subset \cdots \subset M(s_{i_1}\cdot\lambda) \subset M(\lambda).$$

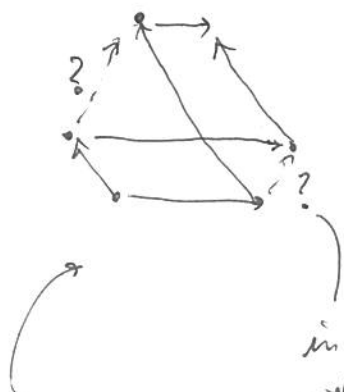
E.g. $\quad w = s_2 s_1 s_2$, $A_2$.

Pf: similar to before, applying $f_i$ for $s_i$.



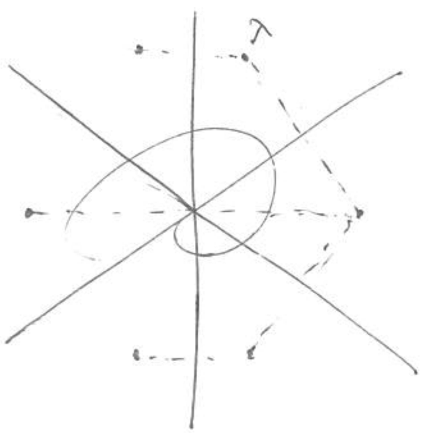

"full" picture of inclusions:

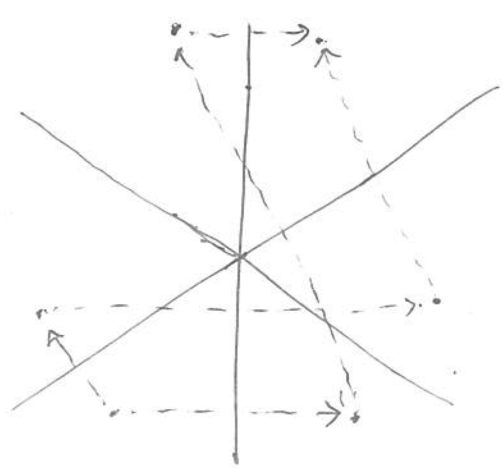



in fact this will be there, but not obvious

(B) see next page



(3) Thm (Verma) — from his thesis

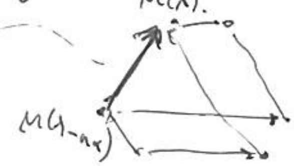
If $\lambda \in \mathfrak{h}^*$, $\alpha \in R_+$ (not nec. simple!).

s.t. $\langle \lambda+\rho, h_\alpha \rangle = n \in \mathbb{Z}_{\geq 0}$,

then $M(\lambda) \supset M(s_\alpha \cdot \lambda)$

$= M(\lambda - n\alpha)$.

(eg in $sl_3$




Pf: technical, not very illuminating ...

see Humphreys, [illegible].