Degeneration of Abelian Differentials and Period Matrices

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Motivations

- The Hodge bundle $\Omega\mathcal{M}_g$ over the moduli of curves $\mathcal{M}_g$ extends to $\overline{\mathcal{M}}_g^{DM}$.
- **Goal: 1)** Study this extension by giving the expansion of an abelian differential in local coordinates near the boundary of $\overline{\mathcal{M}}_g$.
- **2)** To study the degeneration of the period matrices, i.e. gives a description of the boundary of the Torelli image of $\overline{\mathcal{M}}_g$ in $\overline{\mathcal{A}}_g$, the moduli of ppav.

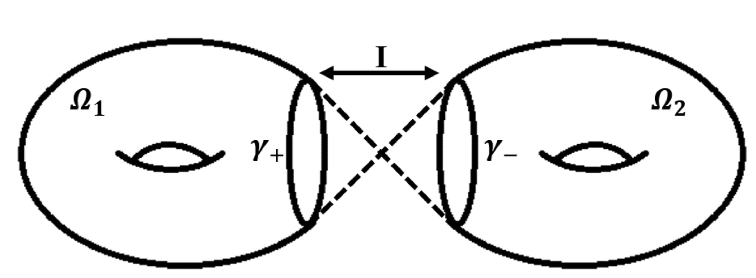
Outline

- We give an explicit expansion in plumbing coordinates for an **arbitrary degenerate** abelian differential.
- We compute the [1]variational formula for the period over **any** given cycle of the abelian differential.
- As a corollary, we give the variational formula for the **period matrices** near an arbitrary boundary stratum of $\overline{\mathcal{M}}_g$.

Plumbing Coordinates

Want: Construct a smooth curve $X_{\underline{s}}$ from a nodal curve X.

- We cut out neighborhoods at the two pre-images q_e and q_{-e} of each node $q_{|e|}$ of X, and identify their boundaries via a gluing map $I_e : z_e \mapsto s_e/z_e$.



- **Before:** Local equation near a node $q_{|e|}$: $z_e \cdot z_{-e} = 0$.
- **After:** Local equation near the *seams* $\gamma_{\pm e}$: $z_e \cdot z_{-e} = s_e$.
- $|s_e|$ is called the **plumbing parameter** at the node $q_{|e|}$.
- The plumbing parameters $\underline{s} := (s_1, \ldots, s_n)$ give **versal deformation coordinates** on $\overline{\mathcal{M}}_g$ to the boundary stratum containing the point X.

Jump Problem

- Given a stable differential Ω on X, denote Ω_v the restriction of a stable differential Ω on the connected component X_v.
- We have the mis-matches $\{\Omega_v(e)|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})\}$ (which we call the **jumps** of Ω) on the seams.
- **Want: Correction differentials $\{\eta_{v,\underline{s}}\}$ that matches the jumps of Ω_v.**
- Then $\Omega_{v,\underline{s}} := \Omega_v - \eta_{v,\underline{s}}$ glue correctly to be a global meromorphic differential $\Omega_{\underline{s}}$ on $X_{\underline{s}}$.
- This construction is called **(solving) the jump problem**. It is first developed and used in a real-analytic setting in [GKN17].

Main Result: Degeneration of Abelian Differentials

- **The solution to the jump problem** $\eta_{v,\underline{s}}$ can be constructed explicitly as $\eta_{v,\underline{s}} = \Sigma_{k=1}^{\infty} \eta_{v,\underline{s}}^{(k)}$.
- The $\underline{s}$-expansion of $\eta_v^{(k)}$ is given as follows:

$$\eta_v^{(k)}(z) = (-1)^k \sum_{l_v^k} \prod_{i=1}^{k} s_{e_i} \cdot \omega_v(z, q_{e_1}) \beta(l_v^k) \operatorname{hol}(\Omega)(q_{-e_k}) + O(|\underline{s}|^{k+1}). \quad (1)$$

- For each k, $||\eta_v^{(k)}||_{L^2}$ is controlled by $\sqrt{|\underline{s}|^k}$. Therefore $||\eta_{v,\underline{s}}||_{L_2}$ is controlled by $\sqrt{|\underline{s}|}$.

Degeneration of general periods

- Let α be any oriented loop on X.
- Let $\{q_1, \ldots, q_N\}$ be the collection of nodes that α passes through (with possible repetition).

Corollary 1 (General Periods)

The variational formula for a general period of $\Omega_{\underline{s}}$ is given by:

$$\int_{\alpha_{\underline{s}}} \Omega_{\underline{s}} = \sum_{i=1}^{N} (r_{e_i} \ln |s_{e_i}| + c_i + l_i) + O(|\underline{s}|^2),$$

here c_i and l_i are the constant and linear terms in $\underline{s}$ respectively, which are explicitly given.

Degeneration of period matrices

- Choose a suitable symplectic basis $\{A_{i,\underline{s}}, B_{i,\underline{s}}\}_{i=1}^g$ of $H_1(X_{\underline{s}}, \mathbb{Z})$.
- Choose a normalized basis of 1-forms $\{v_i\}_{i=1}^g$ w.r.t $\{A_{i,0}, B_{i,0}\}$ on X.
- Apply the jump problem and get $\{v_{i,\underline{s}}\}$ on $X_{\underline{s}}$. We claim that $\{v_{i,\underline{s}}\}$ is a normalized basis of $H^{1,0}(X_{\underline{s}}, \mathbb{C})$ with respect to $\{A_{i,\underline{s}}, B_{i,\underline{s}}\}$.

Corollary 2 (Period Matrices)

For any fixed h, k, the variational formula for the period matrix $\tau_{h,k}(\underline{s})$ is given by

$$\tau_{h,k}(\underline{s}) = \sum_{e \in E_X} N_{|e|,h} \cdot N_{|e|,k} \cdot \ln|s_e| + c_{h,k} + l_{h,k} + O(|\underline{s}|^2) \quad (2)$$

where $N_{|e|,k} := \gamma_{|e|} \times B_{k,\underline{s}}$, E_X is the set of nodes of X, $\{q_{|e_i|}\}_{i=0}^{N-1}$ is the set of nodes B_h passes through. Explicitly,

$$c_{h,k} = \lim_{\underline{s} \to 0} \sum_{i=1}^{N} \left(\int_{p_i}^{p_{i+1}} v_k - N_{|e_i|,h} N_{|e_i|,k} \ln |s_{e_i}| \right)$$

$$l_{h,k} = - \sum_{e \in E_X} s_e \left(\operatorname{hol}(v_k)(q_e) \operatorname{hol}(v_h)(q_{-e}) \right).$$

Prior Works

- Yamada [Yam80] and Fay [Fay73] computed the variational formula of the period matrices on stable curves with one node. We reprove their results by restricting formula (1) to the case $n = 1$.
- For general n, the logarithmic term in formula (2) gives the main result of [Tan91].
- **Our result is a total generalization of these works.**

Compactification of Strata

- Using our method we also give an **alternative proof** for the main result in [BCGGM16], which gives the necessary and sufficient conditions for an abelian differential to lie in the boundary of the *incidence variety compactification* of strata.

Future Works

1. Compute the variational formula for the **period coordinates** on the strata of abelian differentials $\Omega\mathcal{M}_g(\mu)$.
2. Apply the jump problem to a **more general setting**: compute the expansion of a section of any vector bundle as the curve degenerates. For instance, the Higgs bundle.
3. Use the variational formulas of the period matrices of the totally degenerate curve to obtain information about **Teichmüller curves**.

References

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[1] Variational formula = Expansion in terms of s_e and $\ln s_e$.